Network Metrics: A Network Analysis Engine

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Background

Markov-type Lie groups in GL(n,R); Journal of Mathematical Physics; Joseph E. Johnson; Vol. 26, No. 2, February 1985.

Background from our Past Work on Markov Transformations

- Let $x_1, x_2, x_3 \ldots$ be the probabilities for something to be at the locations 1, 2, 3, ….
- Consider the group of transformations, $x(t) = M(t)\, x(t=0)$, that preserves the total probability $\Sigma x_i(t) = \Sigma x_i(0)$
- This is similar to the rotation group which preserves $\Sigma x_i^2(t) = \Sigma x_i^2(0)$

An Example

- Einstein – random walk, diffusion theory 1905
- Markov (1906) transformations (80% probable to stay put and 10% probable to move to the two adjacent cells):

$|X'\rangle = M\,|X\rangle$ can be written as (note columns sum to unity):

$$\begin{pmatrix} 0 \\ 0.1 \\ 0.8 \\ 0.1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.8 & 0.1 & 0 & 0 & 0.1 \\ 0.1 & 0.8 & 0.1 & 0 & 0 \\ 0 & 0.1 & 0.8 & 0.1 & 0 \\ 0 & 0 & 0.1 & 0.8 & 0.1 \\ 0.1 & 0 & 0 & 0.1 & 0.8 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Example of a discrete Markov transformation.
Note the irreversible diffusion.

GL(n,R) – Lie Algebra Decomposition

- Matrices with a ‘1’ at the ij position and a -1 on the diagonal at j,j can be shown to close into a (Markov type) Lie Algebra and form a basis of the off-diagonal matrices.
- Matrices with only a ‘1’ on a diagonal position are a complete basis for the Abelian scaling Lie group.
- All general linear transformations are decomposable into these two Lie groups: Markov type & Abelian scaling.

Define the Lie Algebra for Markov Transformations

- The Markov Lie Group is defined by transformations that preserve the sum of the elements of a vector ie $\Sigma x'_i = \Sigma x_i$
- The generators are defined by ‘rob Peter to pay Paul’ for $M = e^{t\,L}$ where $L = \Sigma_{ij} \eta_{ij} L_{ij}$ is defined by the summation over L with a ‘1’ in row i and column j along with a ‘-1’ in the diagonal position row j and column j
- In two dimensions we get:

$$L^{12} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \qquad L^{21} = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}$$

Lie group defined

- And the Markov group transformation then takes the form:

$$M(t) = e^{s\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}} = \begin{pmatrix} 1 & 1-e^{-s} \\ 0 & e^{-s} \end{pmatrix}$$

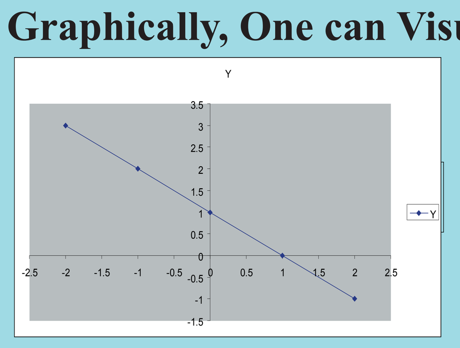
- One notes that the column sums are unity as is required for a Markov transformation.
- This transformation gradually transforms the x_2 value into the x_1 value preserving $x_1 + x_2 = \text{constant}$.

Higher Dimensions

- This Markov Lie Algebra can be defined on spaces of all integral dimensions (2, 3,….) and has $n^2 - n$ generators for n dimensions representing basis elements with a ‘1’ in the ij position and a ‘-1’ in the jj position.
- This makes this basis a complete basis for all off-diagonal matrices. E.g. in 5 dimensions:

$$L^{14} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Graphically, One can Visualize This



- These transformations in two dimensions define the group of motions on a straight line.
- The Markov Lie group is a mapping of this line back into itself – but is NOT a translation group.
- In general one gets hyperplane motions.

Loss of the Markov Inverse giving a Lie monoid

- To be probabilities, non-negative values must be transformed into nonnegative values
- This exactly removes all inverses
- Allowable transformations are those with non-negative linear combinations
- This gives us a Lie Monoid & Irreversibility.

Our New Applications to Networks

I. Networks are here shown to be isomorphic with continuous Markov Monoid transformations. This allows the theory of Lie groups and algebras, as well as Markov processes, to be used to study network topology. These conserved Markov network flows provide a very rich model for understanding network behavior.

II. Generalized Renyi entropy functions can now be defined on the probability distributions that make up the columns of these Markov matrices reducing N values down to one entropy value.

III. These 2N entropy spectra from each row and each column are proposed as the network metrics and their spectra are studied.

IV. Interpretation

I

Networks described

- Networks are defined by off-diagonal non-negative elements.
- A network is totally defined by the flow rates in C

$$C_{ij} = \begin{pmatrix} - & 4 & 2 & 0 \\ 4 & - & 1 & 0 \\ 0 & 0 & - & 1 \\ 1 & 0 & 0 & - \end{pmatrix} \quad \text{modify to:} \quad C_{ij} = \begin{pmatrix} -5 & 4 & 2 & 0 \\ 4 & -4 & 1 & 0 \\ 0 & 0 & -3 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

Network topologies are 1-1 with Markov Lie generators

- The Lie monoid basis for transformations is identical to a connection matrix where the diagonals are set to the negative of the sum of other values.
- Thus network topologies are then 1-1 with Markov Lie generators
- Every possible network corresponds to exactly one Markov family of transformations

II

Entropies can be computed on column probability densities

- Markov matrix columns are probabilities
  Actually this holds to any order of expansion of $M=e^{\lambda C}$ if care is taken to use small λ
- Thus we can compute entropy on each column distributions.
  We prefer to use second order Renyi entropy: $\log_2(N(\Sigma x_i^2))$

Interpretation of Generalized Entropy

- x_i has a high degree of information (order) when it is concentrated in one area
- The integral of the squares is maximum when x_i is concentrated.
- To obtain addativity of independent systems, we define the negative entropy (information) as $I = \log_2(N\Sigma x_i^2)$

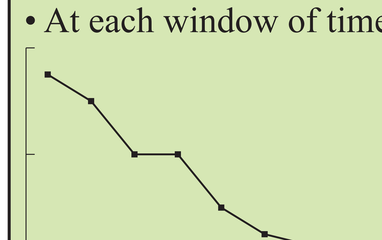
Entropy used as a metric

- The entropy spectra is the order associated with the distributed transfer
- A random uniform distribution to all nodes gives high entropy while an organized transfer to a few nodes give low entropy.
- The entropy function is thus measuring the order associated with transfer process.

III

Spectral curve from sorted entropies

- We only care about the overall pattern
- Thus entropies are sorted in order to derive a spectral curve
- At each window of time, the sorting is redone



Normal & Anomalous Spectra

- Spectral form studied relative to normal
- Correlation to normal studied by many methods such as sums of squares of differences between current and normal entropy
- Two values now represent the network at a given time: row and column sum of squares deviation from the normal
- Wavelets and other expansions can be used on the spectra

IV

Underlying Interpretation and Model

- 1 to 1 correspondence: one network to one family of Markov transformations.
- The interpretation of M is a conserved flow
- Infinitesimal lows are at the rates indicated in C

Entropy Interpretation of the Flows

- The altered C matrix is evolving family of Markov conserved flows corresponding exactly to the topology.
- The entropy encapsulates the order/disorder of that nodes topology.
- The entire spectra captures the entire order and disorder for the entire network

Process

- Network represented by two entropy spectral curves for incoming and outgoing links.
- They represent entropy of a conserved fluid flow on that topology
- Eigenvectors and eigenvalues have clear interpretation.

Applications

- Our software algorithms - now available in Java and Mathematica - are a general purpose tool that can be implemented to study all forms of network dynamics and anomalies.
- We have developed both a totally solid mathematical foundation as well as an intuitive model for interpretation.
- We intend to utilize these algorithms to monitor the behavior of complex networks. Our patents are pending.

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