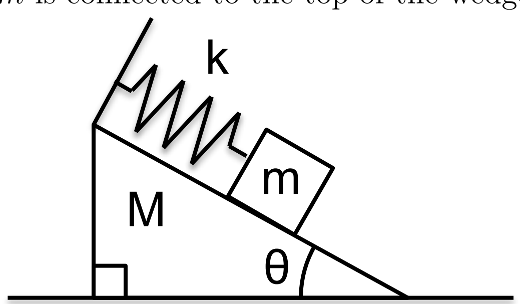
August 2023

CLASSICLAL MECHANICS TEST BANK

1. Carnot Cycle Efficiency [500 level]
   Consider a Carnot cycle using an ideal gas with constant specific heat C_V as the working substance.
   (a) Specify all segments of the path in P-V space that make up the cycle.
   (b) Find the work done on the adiabatic legs.
   (c) Find the work done and heat transfered on the isothermal legs.
   (d) Determine the efficiency.

2. Lagrangian Mechanics [700 level]
   A small object of mass m slides without friction on the inside surface of a cone. The axis of the cone is vertical, with the vertex at the bottom and cone angle θ.
   (a) Find the Lagrangian.
   (b) Find the equations of motion.
   (c) Identify all conserved quantities.

3. Small Oscillations [500 level]
   A mass m slides frictionlessly on the upper surface of a wedge of mass M and angle θ, which in turn slides frictionlessly on a horizontal surface. The mass m is connected to the top of the wedge by a spring, with spring constant k.



   (a) Find the Lagrangian for this system.
   (b) Calculate the frequency of small oscillations about the equilibrium point of the system.

4. Newtonian Mechanics [500 level]
   An object of mass 1 kg moving vertically downward in a uniform gravitational field is subject to a damping force proportional to its velocity (damping force constant 0.5 N/m/s).
   (a) If its initial velocity is forty times its terminal velocity, what is the time

required to reach 1.1 times terminal velocity?
(b) What is the energy dissipated during the slowdown to 1.1 times terminal velocity?

5. Binding Energy [500 level]
(a) How much energy would you have to supply to break up the Earth completely, i.e. convert it into an infinitely dispersed dust? You can ignore: (i) any and all technical hurdles involved in this process (that's for the engineers to worry about); (ii) chemistry and radiation losses; (iii) any nonuniformity in the Earth's composition.
(b) If you reassembled it rapidly (so that no energy radiated, and ignoring chemistry, etc) would it melt?
Some numbers:

$$\begin{array}{rclrcl} M_\oplus &=& 6\times 10^{24}\,\mathrm{kg} & T_{\oplus\mathrm{melt}} &=& 1\times 10^{4}\,\mathrm{K} \\ R_\oplus &=& 6\times 10^{6}\,\mathrm{m} & G &=& 6\times 10^{-11}\,\mathrm{N\,m^2/kg^2} \\ C_\oplus &=& 1\times 10^{3}\,\mathrm{J/kg\,K} & & & \end{array}$$

6. Small Oscillations [500 level]
A pipe of radius r rolls without slipping inside another of radius R. The outer pipe is fixed in position.
(a)Find the frequency of small oscillations in which the axes remain parallel.
(b) Discuss the limits as (i) $r \to 0$ and (ii) $r \to R$.

7. Newtonian Mechanics [500 level]
A satellite attached to the surface of the Earth by a long flexible cable with linear mass density μ, revolves with constant angular velocity ω, equal to the rotation frequency of the Earth.
(a) Show that the cable must be straight and radial.
(b) Derive and solve the differential equation for the tension in the cable as a function of r, the satellite's distance from the center of the Earth. Assume the cable is long compared to the Earth's radius.
(c) Demonstrate that the tension is maximum at

$$r = \left(\frac{gR_\oplus^2}{\omega^2}\right)^{1/3}$$

where g is the acceleration due to gravity at the Earth's surface.

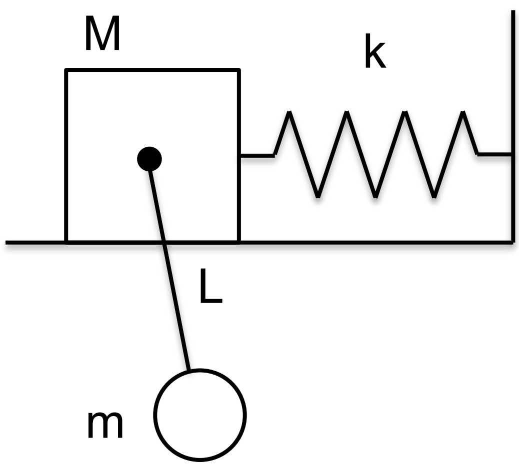
8. Newtonian Mechanics [500 level]
(a) If all linear dimensions of the solar system were doubled, and if the masses of everything in the system were also doubled at the same time, how would the orbital period of the Moon compare to its current value?
(b) By what factor would the total energy change?

9. Hamiltonian Mechanics [700 level]
The motion of an electron in an electromagnetic field is consistent with the Lagrangian

$$L = \frac{1}{2}mv^2 - \frac{e}{c}\mathbf{A} \cdot \mathbf{v} + e\phi.$$

(a) Derive the Hamiltonian.
(b) Show from the Hamiltonian equations of motion that the electron is subject to the Lorentz force.
(c) Solve the equations of motion for the case in which the field is a uniform magnetic field perpendicular to the initial velocity.

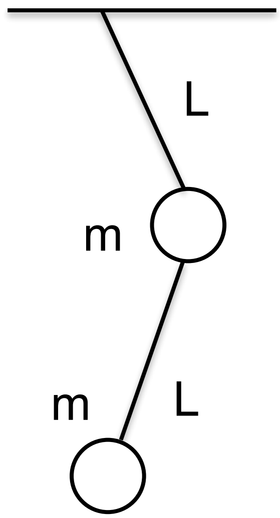
10. Small Oscillations [500 level]
A mass m is suspended, pendulum-like, from another mass M via string of length L. M is attached to a spring of spring constant k and constrained to move only horizontally.



(a) Find the Lagrangian for this system, assuming no friction.
(b) Find the frequency of small oscillation.

11. Statistical Mechanics of a Rubber Band [700 level]
Model a rubber band as a massless chain of N links, each with two possible lengths, L and $2L$. An amount of work W is required to stretch a link from L to $2L$.
(a) Find the entropy of the chain as a function of its length.
(b) If a mass M is suspended from the chain, what is the chain length as a function of temperature T?
(c) What happens in the limits of zero and infinite temperature?
(d) Are there any values of M, W, and L for which the chain contracts for rising temperature?

12. Lagrangian Dynamics [700 level]



(a) For the double pendulum shown, find the kinetic energy in terms of the angular variables and their derivatives.
(b) Deduce the potential energy.
(c) What is the Lagrangian? Find the equations of motion in the angular variables.
(d) Given small oscillations, equal pendula arm lengths, and equal masses, solve the equations of motion.

13. Newtonian Mechanics [500 level]
A rocket initially at rest is fired vertically upward in a uniform gravitational field. The rocket ejects mass, from an initial propellant mass M_p at a muzzle velocity v_p at a rate of α.
(a) Calculate the minimum α required for the rocket to lift off the ground.
(b) For this α, calculate the maximum speed of the rocket.

14. Thermodynamics [500 level]
A house is to be maintained at a temperature lower than the external temperature T_{out}. For convenience, consider a cubical house of side length L. It is cooled by a continuously running air conditioning unit, which is a perfect Carnot engine with constant input power P. The wall thickness is W and their thermal conductivity is k.
(a) Calculate the inside temperature.
(b) What happens when the insulation is poor or the power is low?

15. Action [700 level]
(a) Write down a formula for the period T of a particle executing one-dimensional periodic motion in a potential.

(b) Show that this period T is the derivative of the action

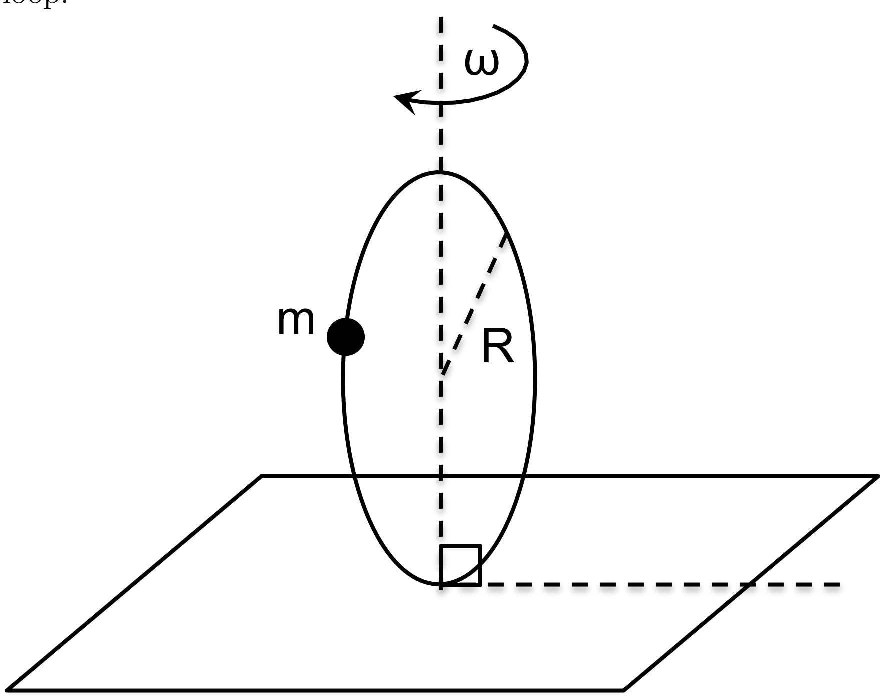
$$\mathcal{I} = \oint dx\, p(x)$$

with respect to the energy E.
(c) Calculate the period of a particle of mass m and energy E in a potential

$$U = \alpha\, |x|\,.$$

16. Ideal Gas [500 level]
A spherical soap bubble with surface tension (energy per unit surface area) Σ is filled with N molecules of an ideal gas. Outside the bubble is vacuum, and you may neglect radiative heat loss.
(a) What is the equilibrium radius of the bubble as a function of temperature?
(b) If the bubble has an initial small radius, $R \approx 0$, how much energy must be supplied in the form of heat to double its radius?

17. Small Oscillations [500 level]
A bead of mass m slides without friction on a circular loop of radius R. The loop lies in a vertical plane and rotates about its vertical diameter with constant angular speed ω. Let θ be the angular position of the bead along the loop.



(a) Show that there is a stable equilibrium at $\theta = 0$ if $\omega < \omega_c$ for some critical

angular speed ω_c, and find ω_c.
(b) Show that if $\omega > \omega_c$, there is a stable equilibrium for some $\theta > 0$ and find that θ as a function of ω.
(c) Find the frequency of small oscillations about $\theta \neq 0$.

18. Newtonian Mechanics [500 level]
On the back of a truck is a cylinder, with radius R and mass M, free to roll. The coefficient of static friction between the cylinder and the truck bed is μ_1. The truck accelerates, causing the cylinder to roll without slipping off the back of the truck. The acceleration is such that, if it were any greater, the cylinder would slip, rather than roll. The cylinder rolls a distance equal to twice its circumference before falling off.
(a) What is the cylinder's angular speed as it leaves the truck?
(b) The coefficient of dynamic friction between the cylinder and road surface is μ_2. How far does the cylinder travel on the ground, Δx, before slipping ceases?
(c) How much work is done on the cylinder by the road? Calculate both the change in kinetic energy of the sphere, W_α and the work done by friction, $W_\beta = F \cdot \Delta x$, and discuss which is right.

19. Lagrangian Dynamics [500 level]
A particle moves in the xy-plane under the influence of a potential $U(x, y) = Kxy$, where K is a constant.
(a) Find the equations of motion.
(b) Find the solution if the particle is initially at rest at $(x_0, 0)$.

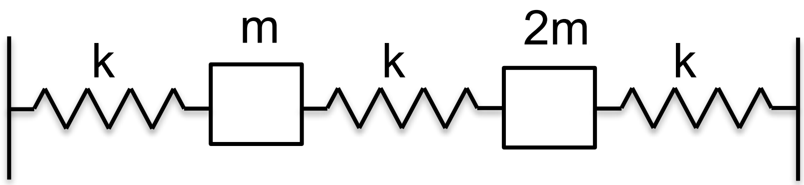
20. Non-Ideal Gas [500 level]
Consider a van der Waals gas, which satisfies the equation of state
$$\left(P + \frac{a}{v^2}\right)(v - b) = kT,$$
where $v = V/N$.
(a) State the first law of thermodynamics and show that C_V, the specific heat at constant volume is
$$C_V = \left.\frac{\partial U}{\partial T}\right|_V$$
(b) State the second law of thermodynamics and show that
$$\left.\frac{\partial U}{\partial V}\right|_T = \lambda_a T \left.\frac{\partial P}{\partial T}\right|_V - \lambda_b P$$
where λ_a and λ_b are constants. Determine λa and λ_b.
(c) Using part (b), find the general form of $U(V, T)$. Using part (a), rewrite $U(V, T)$ in terms of C_V. What simplification results if C_V is constant?

21. Newtonian Mechanics [500 level]
Two particles, m_1 and m_2, attract each other with a central force $\mathbf{F}(r) = -k\mathbf{r}$, where $\mathbf{r}$ is the separation between the particles. Let $V(r = 0) = 0$, with E being the internal energy and L the relative angular momentum.
(a) Given L, what is the allowed range for E?
(b) Given E and L, what are the minimal and maximal distances the particles may be from one-another?

22. Hamiltonian Mechanics [700 level]
Consider a system described by the general coordinates q_i and p_i. If for some conjugate pair q_r and p_r, both q_r and p_r are confined to finite ranges, then the virial theorem holds
$$\left\langle \frac{\partial H}{\partial p_r} p_r \right\rangle = \left\langle q_r \frac{\partial H}{\partial q_r} \right\rangle .$$
You will prove that this is so.
(a) Establish a simple relation between $\frac{d}{dt}(p_r q_r)$ and $\frac{\partial H}{\partial p_r} p_r - q_r \frac{\partial H}{\partial q_r}$.
(b) Consider the average of the relation from (a) over long times, and thereby prove the virial theorem. Recall that
$$\langle f \rangle = \lim_{T \to \infty} \frac{1}{T} \int_0^T dt\, f(t).$$
(c) Use the virial theorem to show that for a three-dimensional harmonic oscillator, $\langle T \rangle = \langle V \rangle$, where T and V are the kinetic and potential energies, respectively.

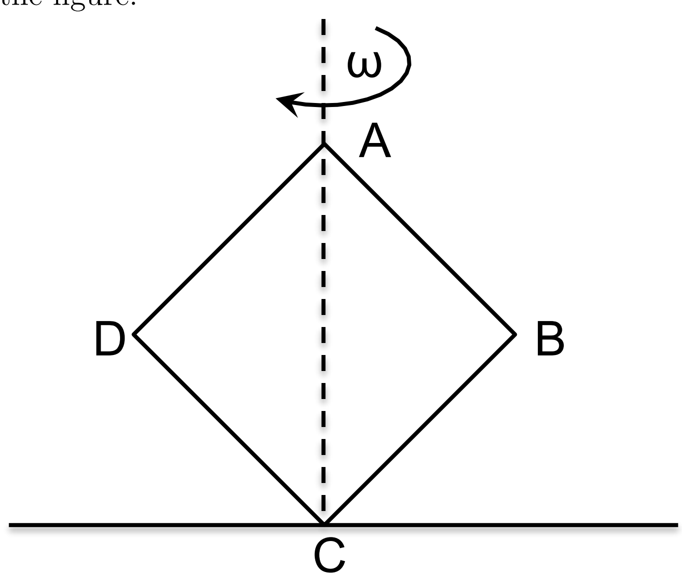
23. Newtonian Mechanics [500 level]
A homogeneous cube of length L is balanced on one edge on a horizontal plane. If the cube is given a slight nudge so that it falls, show that the angular speed of the plane when the face strikes the plane is
$$\omega^2 = \alpha \frac{g}{L}(\sqrt{2} - 1),$$
and find α for the cases of:
(a) no slipping
(b) slipping without friction.

24. Normal Modes [500 level]
Consider the system of masses and springs shown in the figure, with Hookian springs, one-dimensional motion, and no friction.



(a) Find the equations of motion in terms of x_1 and x_2 (the positions of the masses, relative to their equilibrium positions) and find the normal mode frequencies.
(b) Without further calculation:
(i) Give an argument showing that the general motion of the system is a linear combination of the normal modes.
(ii) Describe qualitatively the motions of the two masses in each of the normal modes.
(iii) Describe initial conditions that result in only one normal mode being excited, so that the motion can be described by a single frequency.

25. Newtonian Mechanics [500 level]
A uniform, thin square plate $ABCD$ on a frictionless horizontal surface is rotating around its vertical diagonal AC with angular speed ω, as shown in the figure.



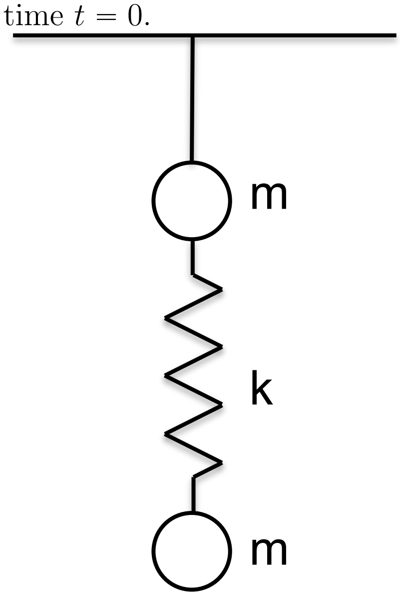
(a) If the vertex B is suddenly stopped, in which direction and with what angular speed does the square rotate around B?
(b) Discuss the conservation (or non-conservation) of mechanical energy in the process given in (a).

26. Hamiltonian Mechanics [500 level]

A particle of mass m is moving in a potential $V = V(r, \theta, \phi)$.
(a) Derive the Hamiltonian of the system in spherical coordinates.
(b) Write down Hamilton's equations of motion.

27. Statistical Mechanics [700 level]
A given system has N distinguishable sites. At each site is a spin that can be in either the up or down state. The spin down state has energy zero, and the spin up state has energy E_0.
(a) What is the partition function for a subsystem with only single site?
(b) What is the partition function for the entire system of N sites?
(c) Derive expressions for the energy E, the free energy F, and the entropy S of the system for the two cases $E_0/kT \gg 1$ and $E_0/kT \ll 1$.
(d) In the high-temperature case $S = Nk \ln 2$. Is this result consistent with $S = k \ln \Omega$, where Ω is the number of accessible states?

28. Newtonian Mechanics [500 level]
Two identical balls of mass m, connected by a spring of constant k, are attached to the ceiling by a string, as shown in the figure. The string is cut at time $t = 0$.



(a) Find the acceleration of each ball at $t = 0$.
(b) Describe the subsequent motion of the balls.

29. Lagrangian Mechanics [500 level]
A small uniform cylindrical mass m, of radius a, sits atop a large fixed cylinder of radius R. The small cylinder is given a small push and rolls without slipping.
(a) Set up the Lagrangian for the system.
(b) When does the small cylinder fall off the large one?

30. Orbital Mechanics [700 level]
(a) Consider an attractive central force $\mathbf{F}(\mathbf{r}) = -(k/r^n)\hat{\mathbf{r}}$. Given k and n, find the condition that guarantees the existence of a stable circular orbit.
(b) Find a similar condition for a general attractive central force $\mathbf{F}(r)$.
(c) Investigate the stability of circular orbits in a force field described by the screened Coulomb potential

$$V(r) = -\frac{k}{r}e^{-r/a},$$

with positive constants k and a.

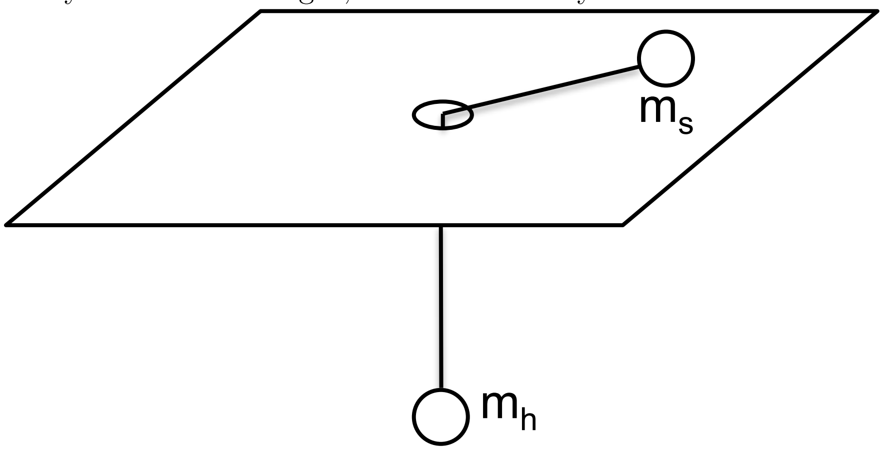
31. Hamiltonian Mechanics [700 level]
(a) For a one-dimensional harmonic oscillator, write down Hamilton's equations of motion and show that they agree with Newton's equations of motion.
(b) Explain the behavior of their solutions in phase space.

32. Statistical Mechanics [700 level]
Consider an ideal gas with N molecules, molecular mass m, volume V, and temperature T.
(a) What are the partition function, the Helmholtz free energy, and the chemical potential for this system?
(b) Let the gas be in contact with a large absorption surface with N_A absorption sites, each of which can absorb at most one molecule. The absorption energy per molecule is E_A. Assuming $N \gg N_A$, what is the grand partition function for the absorption sites?
(c) What fraction of the sites are occupied?

33. Statistical Mechanics [500 level]
Consider a model of the Earth's atmosphere as an ideal gas in equilibrium, at constant temperature in a uniform gravitational field.
(a) Derive an expression relating pressure and mass density of the gas. Identify any constants that you may need, but do not do any numerical calculations.
(b) Derive an expression for the mass density as a function of height above the surface.
(c) Estimate the height above the surface at which the atmosphere is 100 times thinner than at the surface.

34. Lagrangian Mechanics [500 level]
A smooth horizontal plane has a small hole. A string going through the whole connects two masses, m_s and m_h, on the surface, and hanging from the string, respectively. The hanging mass is constrained to move only vertically, while the mass on the surface is constrained to move in the plane. The string is

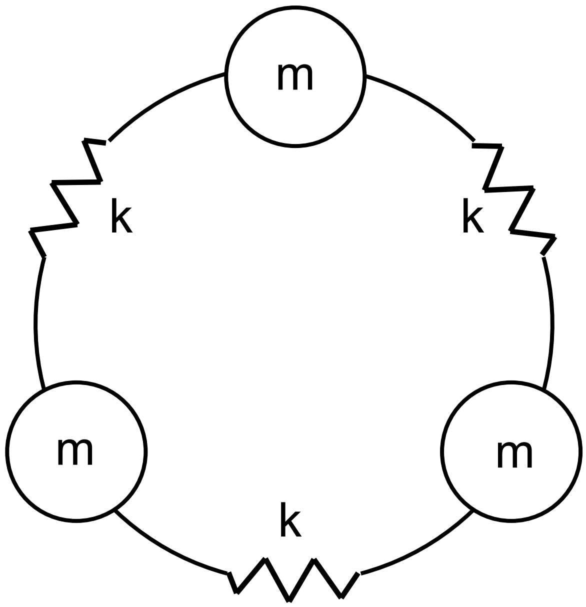
always stretched straight, and the whole system is frictionless.



(a) Find the Lagrangian for this system.
(b) Derive Lagrange's equations of motions.
(c) There are two constants of motion in this system. Identify them.

35. Hamiltonian Mechanics [700 level]
Two masses, m_1 and m_2, move under their mutual gravitational attraction in a uniform external gravitational field whose acceleration is g. Take as generalized coordinates X, Y, and Z for the center of mass (with Z in the direction of g), the distance r between m_1 and m_2, and the angles θ and ϕ which specify the direction of the line from m_1 to m_2.
(a) Write the kinetic energy in terms of the generalized coordinates.
(b) Give expressions for the six generalized momenta.
(c) Give the Hamiltonian for the system.
(d) Find Hamilton's equations of motion.
(w) Identify ignorable (i.e. cyclic) coordinates. Argue that the problem can be reduced to two separate one-degree-of-freedom problems.
(f) Derive expressions for the six generalized forces, $Q_X, Q_Y, \ldots, Q_\phi$.

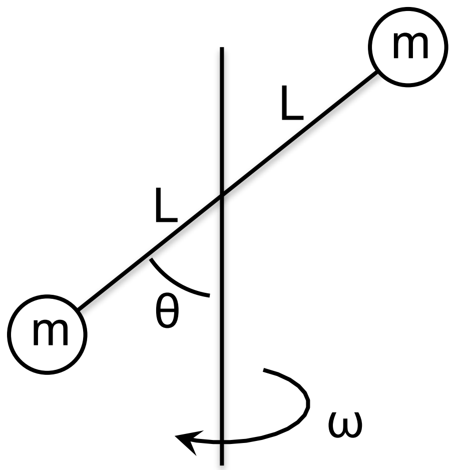
36. Lagrangian Mechanics [700 level]
As shown in the figure, three massless springs, with spring constant k, are connected in a circle with a mass m between each. The system is constrained to move along the circle's circumference.



(a) Find the Lagrangian of the system.
(b) What are the equations of motion for the three displacements, s_1, s_2, and s_3?
(c) Find the normal modes and frequencies, and discuss the characteristics of the normal modes.

37. Moment of Inertia [500 level]
Given a right angle isosceles triangle of uniform planar mass density:
(a) Find the elements of the moment of inertia tensor with respect to a set of coordinate axes which you may choose at your convenience.
(b) What are the principal axes through the center of mass?

38. Hamiltonian Mechanics [700 level]
A massless rod of length $2L$ has identical masses m at each end. The rod is free to rotate about its center, which is attached to a vertical axle. The plane containing the rod and axle is constrained to rotate about the axle with constant angular speed ω.



(a) Find the Lagrangian and Hamiltonian of this two-particle system in terms of one suitable generalized coordinate.
(b) Is H a constant of the motion?
(c) Is the energy of the system conserved?

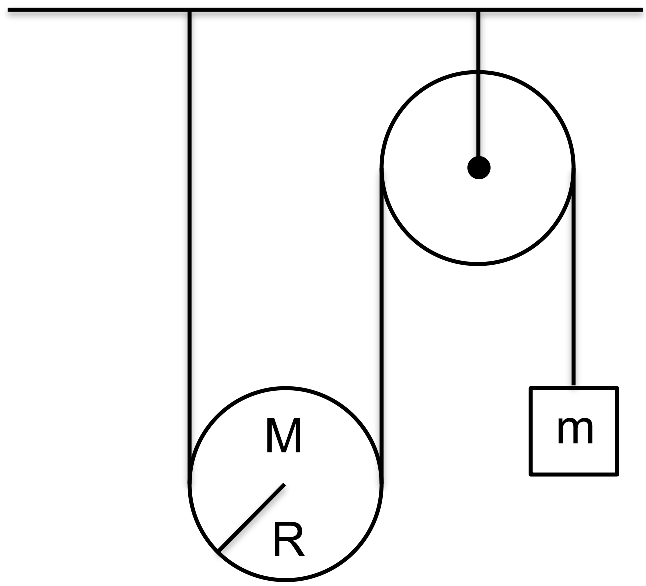
39. Thermodynamics [500 level]
(a) Show that during the adiabatic expansion of an ideal gas, the temperature and pressure changes are related by

$$\frac{dT}{T} = \frac{\gamma - 1}{\gamma}\frac{dP}{P}$$

where the adiabatic index $\gamma = C_P/C_V$ is the ratio of the specific heats.
(b) Assume that air undergoes an ideal gas adiabatic expansion as altitude increases. Derive an expression for the variation of temperature with altitude in terms of the average molecular weight of air and other constants.

40. Newtonian Mechanics [500 level]
A rope of total mass M and length L is suspended vertically with its lower end touching a weighing scale. The rope is released and falls onto the scale.
(a) Find x, the length of rope that has landed on the scale by a time t.
(b) What is the reading of the scale as a function of x?

41. Lagrangian Mechanics [500 level]
Consider the pulley system in the figure. The pulley near the ceiling is fixed and massless. The lower pulley is movable with radius R and mass M.



(a) Obtain the Lagrangian of the system in terms of a suitable generalized coordinate.
(b) Find the equation of motion for this system.
(c) For what values of the parameters does the system have an equilibrium?

42. Lagrangian Mechanics [700 level]
Consider a particle moving in a potential $V(x, y, z)$.
(a) Prove that translational invariance of the Lagrangian in a particular direction implies linear momentum conservation in that direction.
(b) Prove that rotational invariance implies angular momentum conservation.

43. Lagrangian Mechanics [500 level]
The Lagrangian for a spherical pendulum with mass M and rigid rod of length ℓ is

$$L = \frac{1}{2}M(\ell\dot{\theta})^2 + \frac{1}{2}M(\ell \sin\theta\dot{\phi})^2 + Mg\ell\cos\theta.$$

(a) Determine the Lagrangian equations of motion.
(b) Determine the momenta canonically conjugate to θ and ϕ in terms of the coordinates and their time derivatives.
(c) Construct the Hamiltonian.
(d) Determine Hamilton's equations of motion.

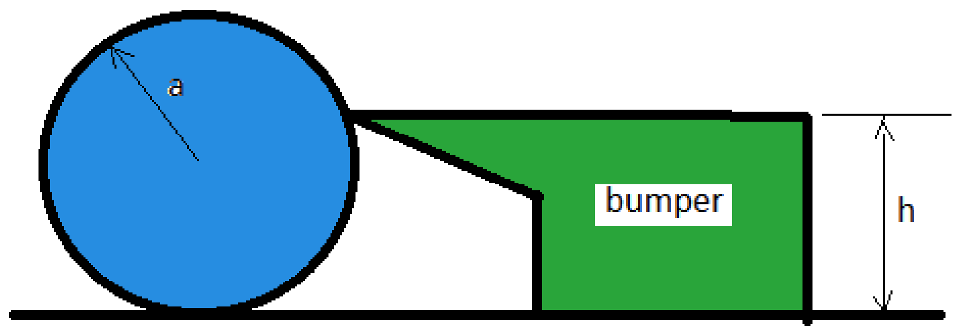
44. Thermodynamics [500 level]
A gas of photons at temperature T in a vessel of volume V has energy $E = aVT^4$ and pressure $P = E/3V$.
(a) Write down the Maxwell relation for the temperature as a derivative of the energy.

(b) Use the result of part (a) and the expression for E to find a differential equation for the entropy S. Solve this equation, noting the undetermined constant.
(c) Infer the value of the constant from the third law of thermodynamics.

45. Newtonian Mechanics [500 level]
A particle moving in one dimension is under the influence of a force $F = -kx + kx^3/a^2$, where k and a are constants and k is positive.
(a) Determine the energy $U(x)$. Assume $U(0) = 0$.
(b) At what other values of x does $U(x) = 0$?
(c) Consider the motion of the particle in the interval $-5|a| < x < 5|a|$. For total energy $E < 0$, describe the possibilities for the trajectory of the particle. Where does the particle go? In what direction does it move? Make a rough plot of $U(x)$ and show the trajectory or trajectories on the plot.
(d) Describe the possibilities for the trajectory for $E > ka^2/4$.
(e) Describe the possibilities for the trajectory for $0 < E < ka^2/4$.

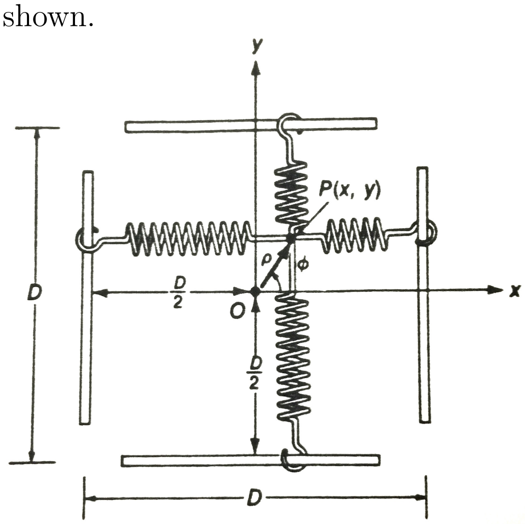
46. Statistical Mechanics [500 level]
A particle of mass m sits on a hot plate with temperature T. The air above the plate is in equilibrium with the plate surface.
(a) Find the average height, $y_0 \equiv \langle y \rangle$, of the particle above the plate due to thermal agitation.
(b) Find the root mean square fluctuation of the average height, $\sqrt{\langle (y - y_0)^2 \rangle}$.

47. Newtonian Mechanics [500 level]
The bumper on a billiard table is designed so that when a ball of radius a, rolling without slipping, strikes the bumper at normal incidence, the ball bounces off without any loss of energy.



(a) What is the torque imparted to the ball during such a collision, if the height of the bumper is h?
(b) Determine h, for an elastic collision with no slipping.

48. Newtonian Mechanics [500 level]
Consider the two-dimensional force due to the four identical, ideal springs with force constant k, each under a small displacement from the equilibrium position. Each spring has one end in common with the other springs and one end that slides frictionlessly on one of four guide rods arranged in a square as shown.



(a) Show that the force on a particle attached to the common point is

$$\mathbf{F} = -2kx\hat{\mathbf{i}} - 2ky\hat{\mathbf{j}}.$$

(b) Is this force conservative? Why?
(c) Show that the potential function is

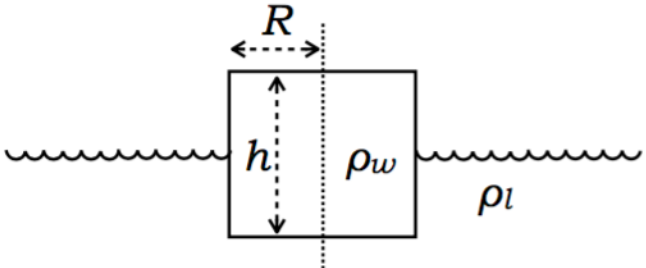
$$U(x, y) = k(x^2 + y^2) + \text{constant}$$

(d) Express $\mathbf{F}$ and U in terms of polar coordinates ρ and ϕ.

49. Thermodynamics [500 level]
A monoatomic ideal gas of N atoms is initially contained in a vessel of volume V and held at temperature T.
(a) The gas is allowed to expand freely to double its volume. What are the temperature, pressure, and internal energy after the expansion?
(b) Then the gas is compressed adiabatically by a piston, back to its initial volume V. What are the temperature, pressure, and internal energy after the compression?
(c) During which of these two processes is entropy generated?

50. Statistical Mechanics [700 level]
Consider an ideal gas, with known number of particles N, pressure P, temperature T, and molecular mass m, contained within a vessel.
(a) Write down the volumetric particle density as a function of particle velocity.
(b) How many impacts per second per unit area are there on the chamber walls?

51. Newtonian Mechanics [500 level]
A cylindrical block of wood of mass density ρ_w, radius R, and height h is partially immersed in a liquid of mass density ρ_ℓ and then released, as shown in the figure.

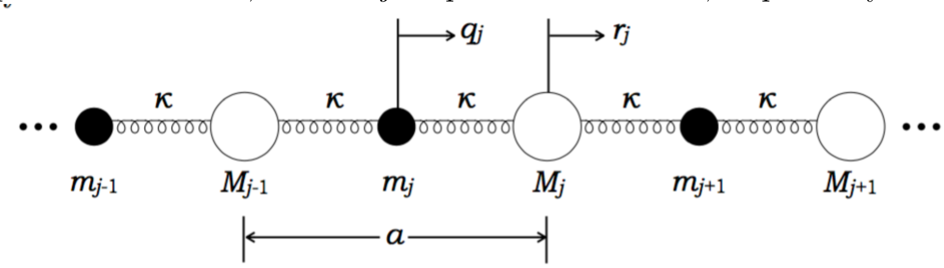


(a) What is the equilibrium height (relative to the top surface of the block) above the water level z_{eq}?

(b) If the block was initially slightly raised, so that $z_0 \equiv z(t=0) > z_{\text{eq}}$, and then released, calculate $z(t)$ assuming no viscosity.
(c) Now assume that the liquid is viscous, and that the viscous force is proportional to the velocity, as given by $F_v = -b\,v$. How is the motion of the block modified? Write down the equation of motion.
(d) What is the condition on the viscous parameter b for the motion to be critically damped?

52. Lagrangian [700 level]
A particle of mass m moves in an attractive central potential $V(r) = -\frac{1}{\alpha}\frac{k}{r^\alpha}$, where k and α are constants. Assume that the angular momentum $\mathbf{L}$ of the particle is not zero.
(a) Write down the Lagrangian. Show that the angular momentum $\mathbf{L}$ of the particle is conserved.
(b) Determine the total energy of the system in terms of m, r, $\dot{r}$, k, α, and $\mathbf{L}$. What is the radial kinetic energy term, which is only a function of the radial velocity of the particle? What is the effective potential energy term $V_{\text{eff}}(r)$, which is a function of the radial position of the particle?
(c) Sketch the effective potential V_{eff} above as a function of r for the three cases $\alpha < 0$, $0 < \alpha < 2$, and $2 < \alpha$. For what values of α does a stable circular orbit exist? For what values of α are all orbits bounded?
(d) For those values of α which support a stable circular orbit, calculate the radius, r_0, of the stable circular orbit in terms of m, r, k, α.

53. Normal Modes [700 level]
Consider an infinitely long, linear harmonic chain with particles of two different masses, m and M, and force constant κ, as shown below. Let a be the equilibrium distance between two neighboring particles of the same mass, and let q_j and r_j be the deviations from their equilibrium positions for the j-th particle of mass m, and the j-th particle of mass M, respectively.



(a) Find the kinetic and potential energies for the system and write down the Lagrangian for the system. Determine the equations of motion for q_j and r_j.
(b) Apply the usual strategy of assuming solutions of the form $q_j = Q_j \exp(i\omega t)$ and $r_j = R_j \exp(i\omega t)$. What are the equations for the amplitudes Q_j and R_j

that result from the equations of motion?
(c) Now, let $Q(k) = \sum_{j=-\infty}^{+\infty} Q_j \exp(ijka)$ and $R(k) = \sum_{j=-\infty}^{+\infty} R_j \exp(ijka)$, where the sum on j is over all particles of mass m for $Q(k)$ or over all particles of mass M for $R(k)$.
Using the above definitions for $Q(k)$ and $R(k)$, which are identified as the normal modes of the system, perform the sum over all of the amplitudes in part b above and obtain the equations for $Q(k)$ and $R(k)$.
(d) Find the normal mode frequencies (i.e. the dispersion relation) $\omega(k)$ for the system. Note that $-\pi/2a \leqslant k \geqslant< \pi/2a$.

54. Thermodynamics [700 level]
Blackbody radiation can be treated as a macroscopic thermodynamic system. Its energy density is given by

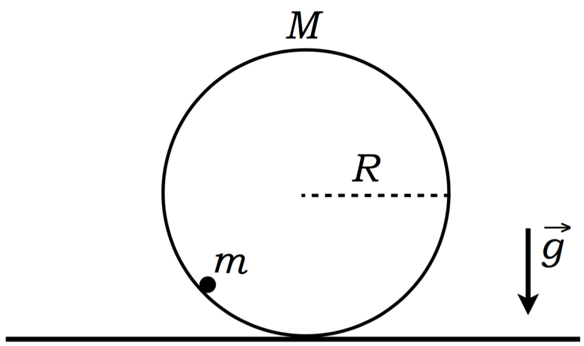
$$U = \frac{4V}{c}\sigma T^4$$

where σ is the Stefan-Boltzmann constant.
(a) Determine the form of the fundamental energy, free energy, or enthalpy relation whose independent variables are V and T, and obtain expressions for the pressure and specific heat. (Note that the entropy S for the system vanishes at $T = 0$.)
(b) Starting from the entropy and internal energy of an ideal gas,

$$\begin{aligned} S &= Nk_B \left\{ \frac{5}{2} + \ln \left[\frac{V}{N} \left(\frac{4\pi m}{3h^2} \frac{U}{N} \right)^{3/2} \right] \right\} \\ U &= \frac{3}{2} Nk_B T, \end{aligned} \tag{1}$$

compute its Helmholtz free energy and the grand canonical potential.

55. Small Oscillations [700 level]
A thin circular loop of radius R, with mass m distributed uniformly along its circumference, is free to roll along a horizontal surface without slipping. A point particle of mass m is attached to the inside of the loop and is constrained to slide along the inside perimeter of the loop without friction. The system is in a uniform gravitational acceleration g.

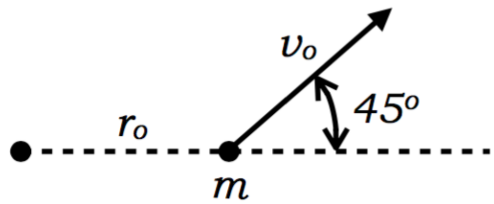


(a) Write down the Lagrangian for this system.
(b) Find the equations of motion and any possible equilibrium positions for the particle.
(c) Which of the equilibrium positions are stable and which are unstable? (You may qualitatively answer this part, if you wish.)
(d) Find the frequency of small amplitude oscillations of the particle about all possible positions of stable equilibrium. Consider your results in the limit $M \gg m$ and discuss whether they are reasonable.

56. Relativistic Kinematics [500 level]
A ball of putty of mass m travels at speed v towards another ball of putty, also of mass m, which is at rest (but which is free to move) in the lab frame. They collide and stick together forming a new object.
(a) Assume that the collision is head on and the speed v of the incoming ball is nonrelativistic, i.e. much smaller than c, the speed of light. Determine the final speed V and mass M of the new nonrelativistic object after the collision.
(b) Again assume that the collision is head on, but now assume that the speed v of the incoming ball is relativistic, so that it can not be neglected compared with c. Determine the final speed V and rest mass M of the new relativistic object after the collision.
(c) Compare the nonrelativistic and relativistic cases with respect to the conservation of total kinetic energy and total mass during the collision. For all cases where there is not conservation, explicitly show non-conservation and qualitatively explain why the value is larger or smaller after the collision.

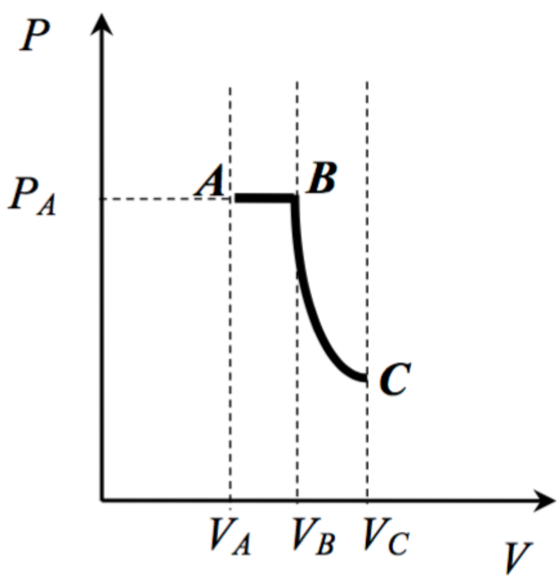
57. Central Force Motion [700 level]
A particle of mass m moves under the influence of the central potential $V(r) = -kr^{-4}$.
(a) Show that the motion occurs in a plane. Hence, use polar coordinates to write the Lagrangian for the system. Determine all constants of the motion.
(b) Make a plot of the effective potential for the radial motion of the particle. Give the general condition for a circular orbit. Does the above potential

support circular orbits? If so, determine them in terms of constants of the motion. Are they stable?
(c) At time $t = 0$ the particle is at $r = r_0$ and is moving with a velocity $v = v_0$ directed at an angle of 45° with respect to the radial outward direction.



Write down the condition for the particle to escape to infinity. From these initial conditions, calculate the minimum value v_0 for which the particle is guaranteed to escape to infinity. You may work in, and leave your answer in, units for which $k = m = 1$.

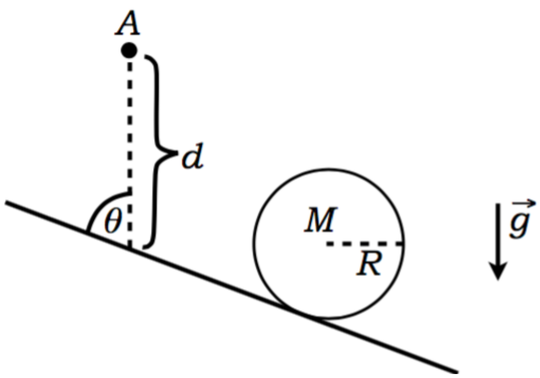
58. Thermodynamics [500 level]
The initial state of monoatomic ideal gas is described by T_A, P_A, and V_A. The gas is taken over the path A→B→C quasistatically as shown in the sketch. The volumes are related: $V_B = \frac{3}{2}V_A$ and $V_C = 2V_A$.



(a) How much work does the gas do on the path A→B and what is the change in its internal energy?
(b) How much heat is absorbed in going from A→B?
(c) Derive the expression for the entropy change for an arbitrary process B→C.
(d) If B→C is an adiabatic process, find the final gas pressure and the entropy change (from the general expression obtained in part c).

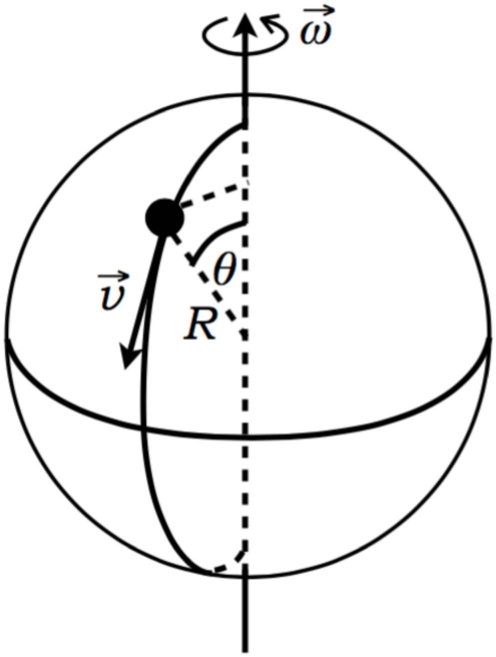
59. Thermodynamics [500 level]
A system of two energy levels E_0 and E_1 is populated by N particles at temperature T. Assume that $E_1 > E_0$.
(a) Derive an expression for the average energy per particle as function of temperature.
(b) Determine the limiting behavior and value for average energy per particle in the limits of zero and infinite temperature.
(c) Derive an expression for specific heat of the system.
(d) Compute specific heat in the limits of zero and infinite temperature.

60. Small Oscillations [700 level]
A homogeneous disk of radius R and mass M rolls without slipping on an inclined surface that makes an angle θ with respect to the vertical. The disk is constrained to be in contact with the inclined plane at all times. The disk is attracted to a point A located at a vertical distance d above the surface.



Assume that the force of attraction is proportional to the distance from the disk's center of mass to the point A; i.e. assume that $F = -kr$, where r is the distance from the point A to the disk's center of mass.
(a) Determine the equilibrium position of the disk, with respect to the position on the surface directly under point A.
(b) Find the frequency of oscillations around the position of equilibrium.

61. Newtonian Mechanics [500 level]
A solid sphere of mass M and radius R rotates freely in space with an angular velocity ω about a fixed diameter. A point particle of mass m, initially at one pole, is constrained to move with a constant speed v on the surface of the sphere and to proceed along a line of longitude (i.e. a great circle).



(a) When the particle has reached the other pole, the rotation of the sphere will have been retarded. Why?
(b) Show that the angle by which the sphere is retarded due to the motion of the particle traveling from one pole to the opposite pole is

$$\Delta\alpha = \omega_0 T \left(1 - \sqrt{\frac{2M}{2M + 5m}}\right),$$

where T is the total time required for the particle to move from one pole to the other and ω_0 is the initial angular velocity of the sphere before the particle begins to move.

You may need to know that

$$\int dx \frac{1}{a^2 + b^2 \sin^2 x} = \frac{1}{a\sqrt{a^2 + b^2}} \tan^{-1}\left(\frac{\sqrt{a^2 + b^2}}{a} \tan x\right).$$

62. Lagrangian Dynamics [700 level]
A particle of mass m is moving under the attraction of an inverse square force of magnitude k/r^2. The particle was initially projected with speed $v = k/2md$ from a point P a distance $2d$ from the center of force O in a direction making an angle $\alpha = \pi/4$ with the line OP.
(a) Determine the energy of the particle, assuming that the potential energy

of the particle at $r = \infty$ is zero.
(b) Determine the angular momentum of the particle.
(c) Determine the minimum and maximum distances of the particle from the force center in the subsequent motion.
(d) Determine the period of the motion.

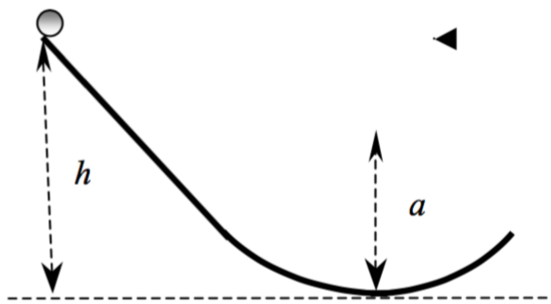
63. Statistical Mechanics [500 level]
An ideal monoatomic gas of N particles, each of mass m, is in thermal equilibrium at temperature T. The gas is confined in a cubical box of side L. The velocities of the particles are distributed according to the Maxwellian distribution.
(a) Deduce the most probable speed $v_{\rm mp}$ of the particles.
(b) Deduce the most probable kinetic energy $\epsilon_{\rm mp}$ of the particles.
(c) What is the average kinetic energy of the particles? No explicit derivation is required.
(d) Now consider the effect of the earth's gravitational field, assumed uniform over the height L of the container, with acceleration due to gravity being g. What is the average potential energy of a particle?

64. Statistical Mechanics [700 level]
DNA can be modeled as two parallel polymer strands with links between the strands called base pairs. Each base pair can be in a closed state with energy 0 or in an open state with energy $\epsilon > 0$. Consider a DNA molecule with N base pairs in thermal equilibrium at temperature T. Thermal fluctuations can cause each base pair to open, leading to separation of the two strands. Assume that the two strands are tethered together at the right end such that the molecule can open only from the left end, and only in sequential order (i.e. base pair s can open only if $1, 2, \ldots, s-1$ to the left of it are already open).
(a) Show that the partition function, Z, for this system is given by the following expression:

$$Z = \frac{1 - \exp\left[-\frac{(N+1)\epsilon}{k_B T}\right]}{1 - \exp\left(-\frac{\epsilon}{k_B T}\right)}.$$

(b) In the limit $N \to \infty$, determine the mean number $\langle n \rangle$ of open base pairs.
(c) Evaluate $\langle n \rangle$ from part b in the limit of zero and infinite temperature.
(d) Next, consider the same DNA molecule now surrounded by a protein P at concentration c. Protein P can bind to the DNA only at a site that is open. Assume each protein P can occupy no more than one base pair. The chemical potential for P is $\mu = \mu_0 + k_B T \ln(c/c_0)$, where $c_0 > 0$ and μ_0 are constants. Write down a closed form expression for the Grand canonical ensemble for this system, in the limit of infinite number of base pairs.

65. Newtonian Mechanics [500 level]
A cylinder of a non-uniform radial density with mass M, length L, and radius R, rolls, without slipping, from rest down a ramp and onto a circular loop of radius a. The cylinder is initially at a height h above the bottom of the loop. At the bottom of the loop, the normal force on the cylinder is twice its weight.



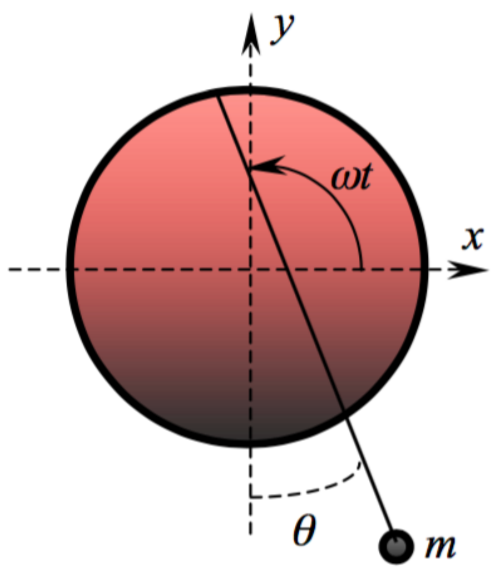
(a) Expressing the rotational inertia of the non-uniform cylinder in the general form $I = \beta M R^2$, determine β from h and a.
(b) Find numerical value of β if the radial density profile for the cylinder is given by $\rho(r) = Ar^2$.
(c) If the radial density profile is given by $\rho_n(r) = Br^n$, where n is a positive integer, describe qualitatively how do you expect the value of β to change with increasing n. Explain your reasoning.

66. Orbital Mechanics [700 level]
A particle of unit mass is projected with a velocity v_0 at right angles to the radius vector at a distance a from the origin of a center of attractive force, given by

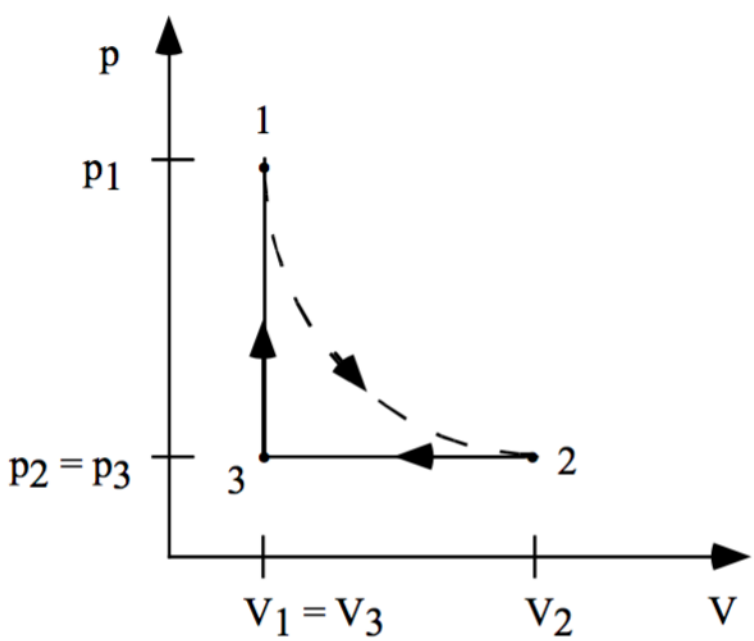
$$f(r) = -k\left(\frac{4}{r^3} + \frac{a^2}{r^5}\right).$$

For initial velocity $v_0^2 = 9/2a^2$, find the polar equation of the resulting orbit.

67. Lagrangian Mechanics [700 level]
A simple pendulum of length b and mass m is suspended from a point on the circumference of a thin massless disc of radius a that rotates with a constant angular velocity ω about its central axis. Using Lagrangian formalism, find
(a) the equation of motion of the mass m;
(b) the solution of the equation of motion for small oscillations.



68. Thermodynamics [500 level]
Suppose one mole of an ideal gas is subjected to the cyclic process shown below (with volume V_1, V_2, and V_3 in states 1, 2, and 3, respectively).



$1 \rightarrow 2$ is an isothermal expansion. $2 \rightarrow 3$ is an isobaric expansion. $3 \rightarrow 1$ is an isochoric heating step. All steps are reversible.
(a) What is the change in internal energy, ΔU, for the entire cyclic process $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$?

(b) Use the First Law of Thermodynamics to calculate ΔU, δQ, and δW for the process $2 \to 3$.
(c) Use the First Law of Thermodynamics to calculate ΔU, δQ, and δW for the process $3 \to 1$.
(d) Is the total work done in a cycle positive or negative? What is the efficiency, η of this cycle? In which limit does one obtain $\eta = 1$?

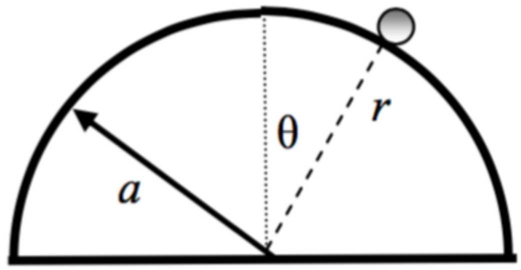
69. Statistical Mechanics [700 level]
Consider a system consisting of M non-interacting molecules at temperature T. Each of these molecules possesses vibrations with energies

$$E_n = \hbar\omega_0 \left(n + \frac{1}{2}\right)$$

where $n \in \{1, 2, \ldots, N_0\}$. First consider the case of $N_0 = \infty$.
(a) Using the partition function, compute the total energy, $\langle E \rangle$, of the system for the $T \to 0$ and $T \to \infty$ limits. Explain your results. At what temperature does $\langle E \rangle$ change from the $T \approx 0$ to the $T \approx \infty$ type?
(b) Compute $\langle n \rangle$ for $T \to \infty$. What is the physical interpretation of $\langle n \rangle$? What is the relation of $\langle n \rangle$ to the partition function and to $\langle E \rangle$?
(c) Let $N_0 \in \mathbf{Z}^+$ be finite. What is the form of $\langle E \rangle$ when the temperature $T \to \infty$?
(d) Compute the specific heat, C_V, of the system in the limit $T \to \infty$ for the two cases $N_0 = \infty$ and $N_0 < \infty$. Explain the difference in C_V between these two cases.

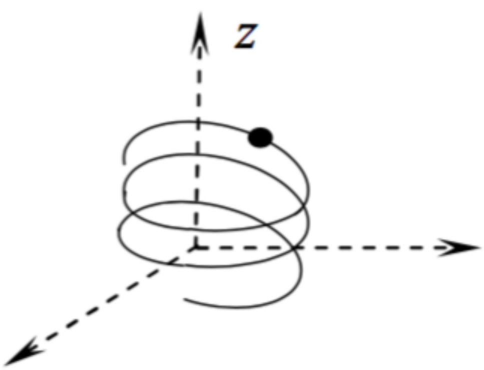
70. Thermodynamics [500 level]
Consider a monoatomic ideal gas.
(a) What is the internal energy and the equation of state of an ideal gas?
(b) Compute the entropy of an ideal gas as a function of T and V for constant particle number N starting from $dU = TdS - PdV$.
(c) Find the chemical potential as a function of P and T starting from the Gibbs-Duhem relation, $SdT - VdP + Nd\mu = 0$.

71. Newtonian Mechanics [500 level]
The particle is sliding down from the top of the hemisphere of radius a.
(a) Find the normal force exerted by the hemisphere on the particle.
(b) Find the angle with respect to the vertical at which the particle will leave



the hemisphere.

72. Hamiltonian Mechanics [700 level]
A particle of mass m moves frictionlessly under the influence of gravity along the helix $z = k\theta$, $r = r_0$, where k and r_0 are constants and z is vertical.
(a) Find the Lagrangian.
(b) Find the Hamiltonian.
(c) Determine the equations of motion.



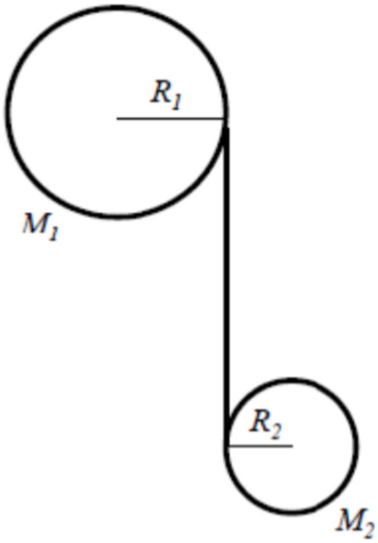
73. Newtonian Mechanics [500 level]
A particle of mass m is bound by the linear potential $U = kr$, where k is a constant.
(a) For what pairs of energy and angular momentum will the orbit be a circle of radius r about the origin?
(b) What is the frequency of this circular motion?
(c) If the particle is slightly disturbed from this circular motion, what will be the frequency of small oscillations?

74. Statistical Mechanics [500 level]
Consider a spherical drop of liquid water containing N_1 molecules surrounded by $N - N_1$ molecules of water vapor. The drop and its vapor may be out of equilibrium.

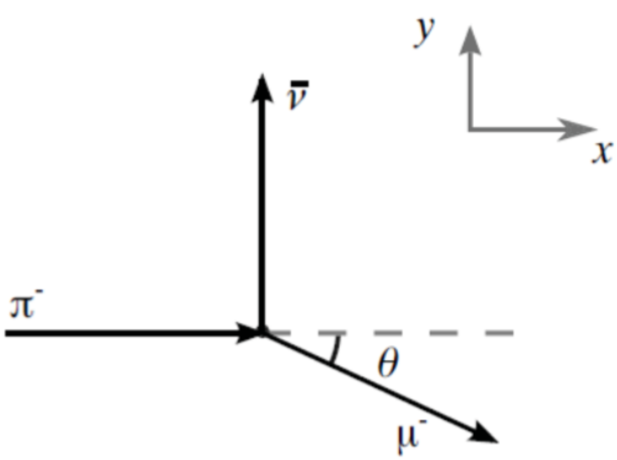
(a) Neglecting surface effects write an expression for the Gibbs free energy of this system if the chemical potential of liquid water in the drop is μ_l and the chemical potential of water in the vapor is μ_v. Rewrite N_1 in terms of the (constant) volume per molecule in the liquid, v_l, and the radius r of the drop.
(b) The effect of the surface of the drop can be included by adding $G_{\text{surf}} = \sigma A$ to the free energy, where $\sigma > 0$ is the surface tension and A is the surface area of the drop. Write G_{tot} with the surface component expressed in terms of r. Make two qualitative hand-sketches of G_{tot}: one sketch with $(\mu_l - \mu_v) > 0$ and one sketch with $(\mu_l - \mu_v) < 0$. Describe the behavior of the drop in these two cases.
(c) Under appropriate conditions there is a critical radius, r_c, that separates drops which grow in size from those that shrink. Determine this critical radius.

75. Statistical Mechanics [500 level]
A gas of N identical classical non-interacting atoms is held in a potential $V(r) = ar$, where $r = \sqrt{x^2 + y^2 + z^2}$. The gas is in thermal equilibrium at temperature T.
(a) Find the single particle partition function Z_1 of an atom in the gas. Express your answer in the form $Z_1 = AT^{\alpha}a^{-\eta}$, providing expressions for the prefactor exponents α and η.
(Hint: convert the integral in r to spherical coordinates.)
(b) Find an expression for the entropy S of this classical gas.

76. Newtonian Mechanics [500 level]
A frictionless pulley, constructed from a solid disk of mass M_1 and radius R_1, can rotate about its horizontal axis of rotation. A string is wound around the pulley, with its other end wound around a second pulley of mass M_2 and radius R_2 that is falling downwards while maintaining the horizontal orientation of its axis. Assume that the string is massless, does not slip, and remains vertical and taut during the motion.
(a) Find the linear acceleration of the center of mass of M_2.
(b) Find the angular acceleration of M_2.
(c) Find the angular acceleration of M_1.
(d) Find the tension in the string.

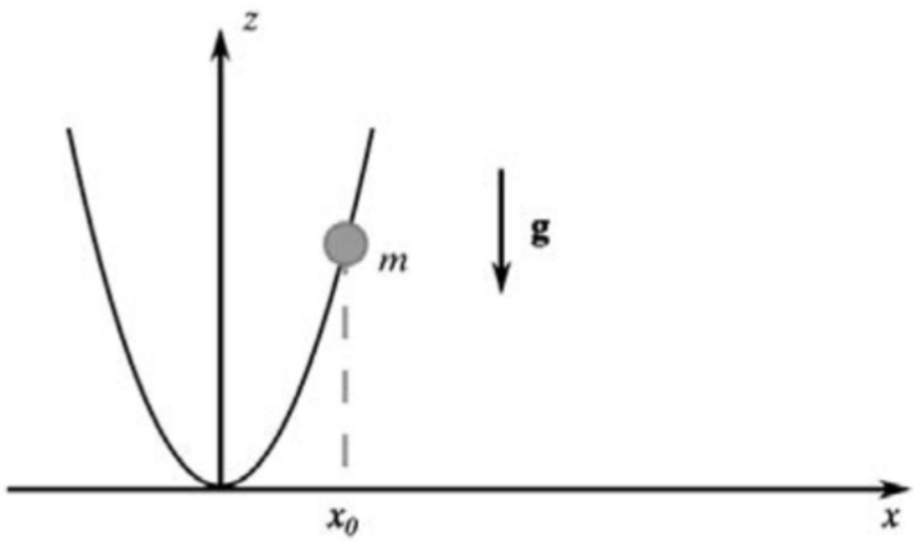


77. Special Relativity [500 level]
(a) A pion π^- is moving with a velocity v and decays into a muon μ^- and a massless antineutrino $\bar{\nu}_\mu$. If the antineutrino moves after the decay perpendicular to the direction of the π^-, find the energy of the muon and the angle θ of the muon's direction relative to the π^- in terms of: the π^- mass m_π, the muon mass m_μ, $\beta_\pi = v/c$, and $\gamma_\pi = 1/\sqrt{1-\beta_\pi^2}$.
(b) If the π^- is at rest when it decays into a muon and an antineutrino, find the distance traveled by the muon before it decays (i.e., during its lifetime τ) in terms of m_π, m_μ, and the muon lifetime τ.



78. Lagrangian Mechanics [700 level]
A bead of mass m slides under gravity along a smooth vertical parabolic wire.

The shape of the wire is given by the equation $ax^2 - z = 0$. The bead starts from rest at $x = x_0$.
(a) Write the Lagrangian of the system.
(b) Use the Lagrange multiplier method to determine the force that the wire exerts on the bead as a function of x.



79. Lagrangian Mechanics [700 level]
A point particle of mass m is moving under the potential

$$V(x, y) = \frac{k}{2} \sin^2(\sqrt{x^2 - xy + y^2}/a)$$

where k and a are positive constants.
(a) Write the Lagrangian of the system.
(b) Prove that the origin $x = y = 0$ is a stable equilibrium point and write the Lagrangian appropriate for small oscillations about this point.
(c) Find the normal frequencies of the system.
(d) Construct the normal coordinates of the system and express the Lagrangian in terms of these coordinates.

80. Newtonian Mechanics [500 level]
An object of mass m is thrown vertically upward from the earth's surface with initial speed v_0. There are only two forces acting on the object: its weight and the air resistance which is opposite to the direction of motion and has a magnitude of kmv^2, where k is a positive constant and v is the object's speed at time t.
(a) Find the maximum height H reached by the mass as a function of k, v_0,

and the acceleration of gravity g.
(b) After the mass has reached the maximum height H, it starts falling down. If $kv_0^2 < g$, find the distance, as a function of k and H, the mass has dropped from its maximum height when it reaches speed v_0.

81. Thermodynamics [500 level]
A mole of fluid obeys the van der Waals equation of state,

$$P = \frac{cT}{V-b} - \frac{a}{V^2},$$

where a, b, and c are positive constants.
(a) Show that the heat capacity at constant volume (C_V) is a function of T only.
(b) Is this result still true if the fluid obeys the Dieterici equation of state

$$P = \frac{cT}{V-b} e^{-\frac{1}{VRT}}?$$

(c) Find the entropy $S(T, V)$ of the van der Waals fluid in the case when C_V is independent of T.

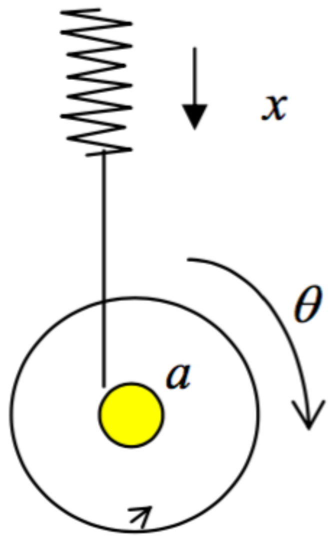
82. Thermodynamics [500 level]
One mole of ideal gas with a constant heat capacity C_V is placed at a pressure P_1 inside a cylinder, which is thermally insulated from the environment. The volume of the cylinder can be changed using a piston, which moves without friction along the piston axis.

At the beginning of the experiment, the pressure of the gas is abruptly changed from P_1 to P_2, resulting in a decrease in the volume from V_1 at pressure P_1 to V_2 at pressure P_2. Similarly, the temperature T_1 will increase to T_2 after thermodynamic equilibrium has been reached.
(a) Determine the type of compression that took place here. Explain.
(b) Find the T_2 and V_2 in terms of V_1, P_2 and T_1 after thermodynamic equilibrium has been reached. Hint: Use the equation of state for an ideal gas and the relationship between C_V and C_P.
(c) After thermodynamic equilibrium has been established (after the pressure changes to P_2), the pressure is abruptly reset to its original value, resulting in V_f and T_f. Find the final values of T_f and V_f in terms of P_1, V_2, and T_2 after thermodynamic equilibrium is reached again. Use the First Law of Thermodynamics and the adiabatic equation to compute the difference in the temperatures $T_f - T_1$. Comment on both the sign and the relative magnitude of the temperature difference. What happens in the limit of very small changes in pressure?

83. Hamiltonian Mechanics [700 level]
A point-like mass m is undergoing a three-dimensional projectile motion in a uniform gravitational field g. Air resistance is negligible.
(a) Write the Hamiltonian function for this problem, choosing as generalized coordinates the Cartesian coordinates (x, y, z) with the z axis pointing in the vertical direction.
(b) Show that these equations lead to the known equations of motion for projectile motion and that the Hamiltonian function corresponds to the total mechanical energy of the particle.
(c) Once the particle reaches the highest point in its trajectory, a retarding force proportional to the velocity of the particle starts acting. Assume the proportionality constant to be known and given by k. Calculate the vertical velocity as a function of time for the descending particle, and find the terminal velocity.

84. Newtonian Mechanics [500 level]
Consider the one-dimensional motion of a rocket in outer space. The rocket is not subject to the influence of any external force, but rather moves by the reaction of ejecting mass at high velocities. At some arbitrary time t, the instantaneous total mass of the ship is $m(t)$, and its instantaneous velocity with respect to an inertial reference system is $v(t)$. During a time interval dt, a mass $-dm$ is ejected from the rocket engine with a speed $-u$ with respect to the rocket. Consider that the rate at which mass is ejected from the rocket is constant and given by μ. How far will the rocket have traveled once it lost half of its initial mass? Assume at the initial time the rocket's total mass is m_0, and it started from rest. Suppose the velocities involved are small enough to allow you to treat the problem with non-relativistic mechanics.

85. Lagrangian Mechanics [700 level]
A yo-yo of mass m and rotational inertia I rolls down due to gravity. The end of its string is attached to a spring of negligible mass and spring constant K. The radius of the axle of the yo-yo is a, as indicated in the figure. Let x be the extension of the spring measured with respect to its natural length.



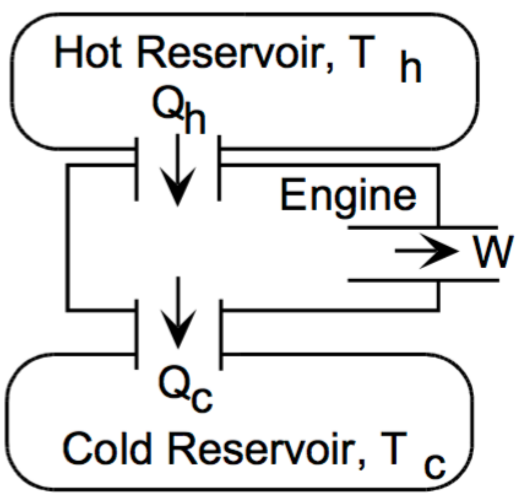
(a) Using the generalized coordinates x and θ indicated in the figure, write the Lagrange equations of motion for the yo-yo.
(b) Find the oscillation frequency of the spring while the yo-yo is rolling down.
(c) Consider the limit of a thin axle ($ma^2 << I$) and solve the differential equation for x you found in part b. Explain the motion described by your solution.

86. Thermodynamics [500 level]
A vertical cylinder contains n moles of an ideal gas, and is closed off by a piston of mass m and area A. The acceleration due to gravity is g. The molar specific heat C_V at constant volume of the gas is a constant independent of temperature. The heat capacities of the piston and the cylinder are negligibly small, and any frictional forces between the piston and the cylinder walls can be neglected. The whole system is thermally isolated. Assume that the external air pressure is negligible.

Initially the piston is clamped in position so that the gas has a volume V_0 and a temperature T_0. The piston is then released and, after some oscillations, comes to rest in a final equilibrium state corresponding to a larger volume of the gas.
(a) Does the temperature of the gas increase, decrease, or stay the same? Explain your answer.
(b) Does the entropy of the gas increase, decrease, or remain the same? Explain your answer.
(c) What is the value of the ratio T_f/V_f, where T_f and V_f are the final temperature and volume of the gas?
(d) What is the net work done by the gas in this process?

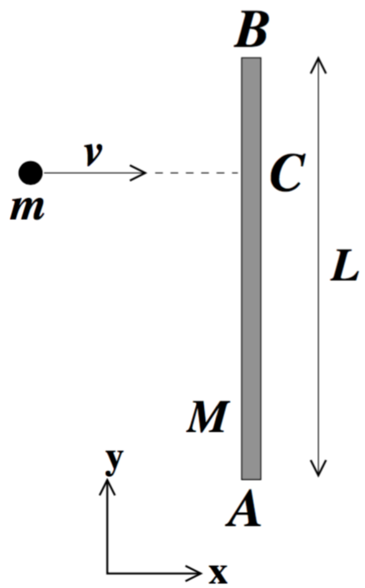
(e) What is T_f?

87. Thermodynamics [500 level]
The diagram below shows the energy flow in a heat engine.
(a) State the definition of the efficiency of this heat engine in terms of the heat Q_h absorbed from the hot reservoir, and the heat Q_c rejected to the cold reservoir over one cycle. Do not assume that the engine is reversible.
(b) Based upon entropy considerations, derive an inequality between the efficiency of the engine, and the temperatures of the reservoirs.
For parts c–e, replace the reservoirs by two identical but finite bodies, each characterized by a heat capacity at constant volume C_V, which is independent of temperature. The bodies remain at constant pressure and undergo no change of phase. Initially, their temperatures are T_1 and T_2, respectively; finally, as a result of the operation of the heat engine, the bodies will attain a common final temperature T_f.
(c) What is the total amount of work done by the engine? Do not assume that the engine is operated reversibly.
(d) As in part b, use entropy considerations to derive an inequality relating T_f to the initial temperatures T_1 and T_2.
(e) For given initial temperatures T_1 and T_2, what is the maximum amount of work obtainable by operating an engine between these two bodies?

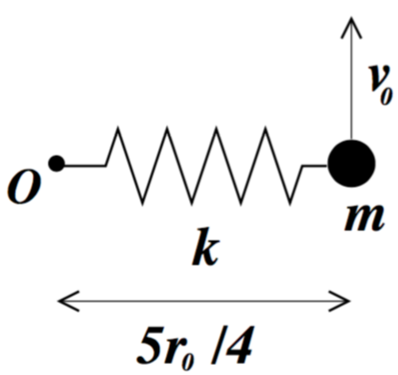


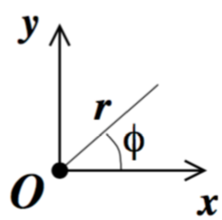
88. Newtonian Mechanics [500 level]
A thin uniform rod of mass M and length L lies on a horizontal surface, aligned along the y direction as shown below. An object of mass m moving along the x direction with a speed v collides with the rod at a point C.
At what point should the object hit the rod so that, immediately after the collision, the rod has an instantaneous axis of pure rotation (i.e. it is only

rotating, not translating) around point A?



89. Lagrangian Mechanics [500 level]
A massless spring of force constant k and natural length r_0 lies on a horizontal frictionless table. The spring is attached to the table at one end (the origin O), and can rotate freely around it. An object of mass m is attached to the other end of the spring.

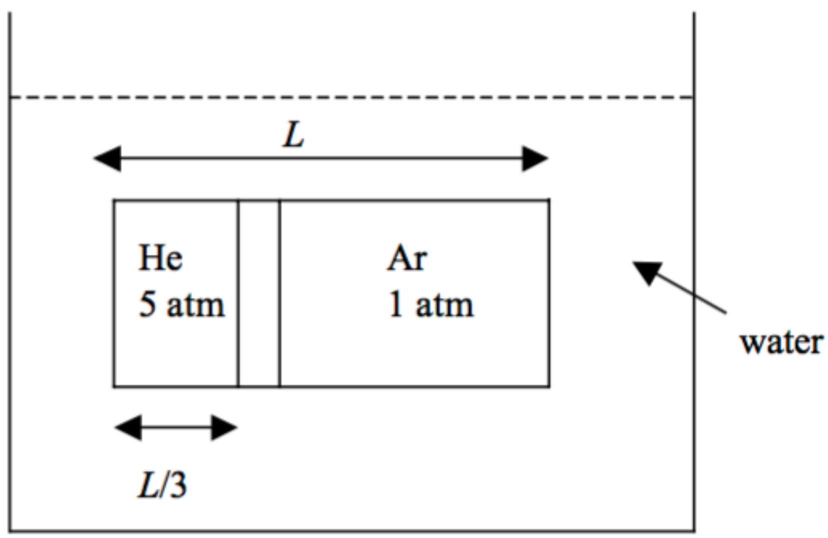




(a) Assume that the spring is initially stretched along the x axis to a length of $\frac{5}{4}r_0$, and the object is given an initial speed of v_0 in the $+y$ direction. At what value of v_0 will the object rotate in a circle of fixed radius $\frac{5}{4}r_0$ about the origin?
(b) Using the polar coordinate system (r, ϕ), construct the Lagrangian and:
(i) Identify the cyclic coordinate and interpret the associated conserved quantity.

(ii) Show that the total energy of the system can be expressed as $\frac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r)$.
(iii) Sketch $V_{\text{eff}}(r)$ and explain why there exists a stable circular orbit.
(iv) If the radius of the stable circular orbit is $\frac{5}{4}r_0$, show that the magnitude of the conserved quantity is consistent with the result obtained in part a.
(c) As the object moves in a fixed circle of radius $\frac{5}{4}r_0$, it is given a small additional push in the radial direction. Calculate the frequency of the resulting small radial oscillations in terms of k and m only.

90. Thermodynamics [500 level]
A cylindrical container of length L is separated into two compartments by a thin piston, originally clamped at a position $L/3$ from the left end. The left compartment is filled with 1 mole of helium gas at 5 atm of pressure; the right compartment is filled with argon gas at 1 atm of pressure. These gases may be considered ideal. The cylinder is submerged in 1 liter of water, and the entire system is initially at the uniform temperature of 25° C, and thermally isolated from the surroundings. The heat capacities of the cylinder and the piston may be neglected. When the piston is unclamped, the system ultimately reaches a new equilibrium situation.

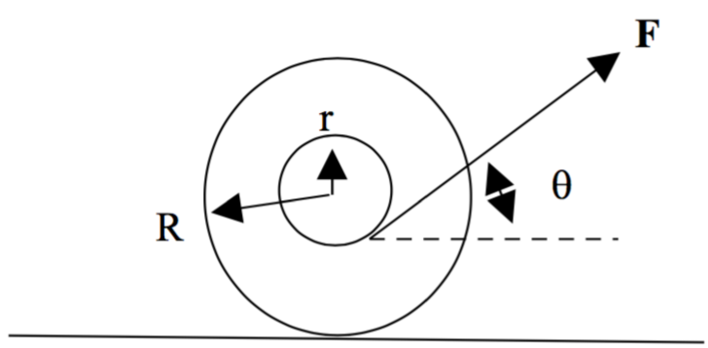


(a) What is the change in the temperature of the water?
(b) How far from the left end of the cylinder will the piston come to rest?
(c) Starting from $dS = \left(\frac{\partial S}{\partial V}\right)_T dV + \left(\frac{\partial S}{\partial T}\right)_V dT$, find the total increase in the entropy of the system.
(d) Now consider a slightly different situation, in which the left side of the cylinder contains 5 moles of real (not ideal) gas, with attractive intermolecular

interactions. The right side still contains 1 mole of an ideal gas. As before, the piston is initially clamped at a position $L/3$ from the left end. When the piston is unclamped and released, does the temperature of the water increase, decrease, or stay the same? Does the internal energy of the gas increase, decrease, or remain the same? Explain your reasoning.

91. Thermodynamics [500 level]
A copper block is cooled from T_B to T_A using a Carnot engine operating in reverse between a reservoir at T_C and the copper block. The copper block is then heated back up to T_B by placing it in thermal contact with another reservoir at T_B ($T_C > T_B > T_A$).
(a) What is the limiting value of the heat capacity per mole for the copper block at high temperatures?
(b) Find the total entropy change of the universe in the cyclic process B→A→B, and show that it is greater than zero.
(c) How much work is done on the system, consisting of the copper block and the Carnot engine?
(d) For the cyclic path B→A→B, does the system absorb heat from the reservoirs or reject heat?

92. Newtonian Mechanics [500 level]
A toy consists of two concentric cylinders with inner radius r and outer radius R, rigidly attached to one-another. A string is wound around the inner radius, and the outer radius can roll without slipping on a rough floor. The string is pulled at angle θ with respect to the horizontal.

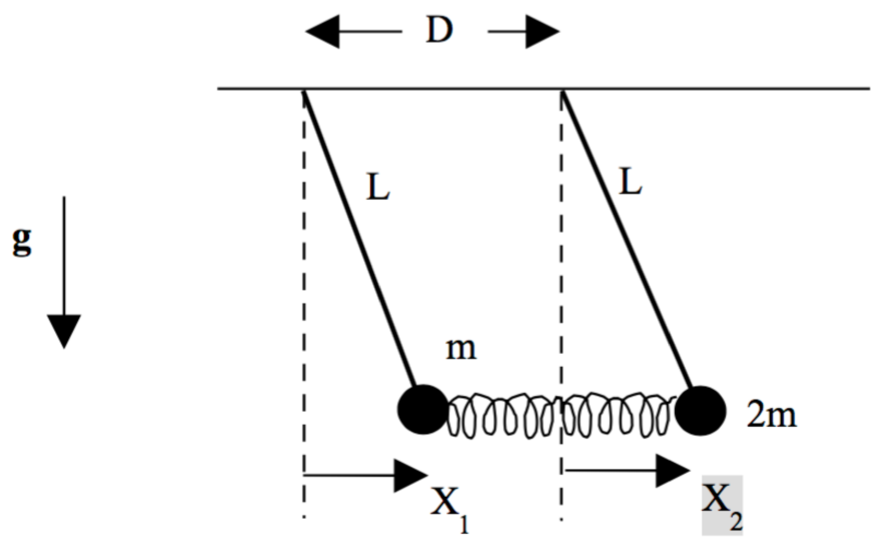


(a) Draw the free body diagram.
(b) Calculate the angular acceleration.
(c) Prove that there exists a critical angle θ_c, where if $\theta < \theta_c$ the cylinder rolls away from the direction it is pulled, and if $\theta < \theta_c$ the cylinder rolls toward the direction it is pulled.
(d) Determine θ_c.

93. Special Relativity [500 level]

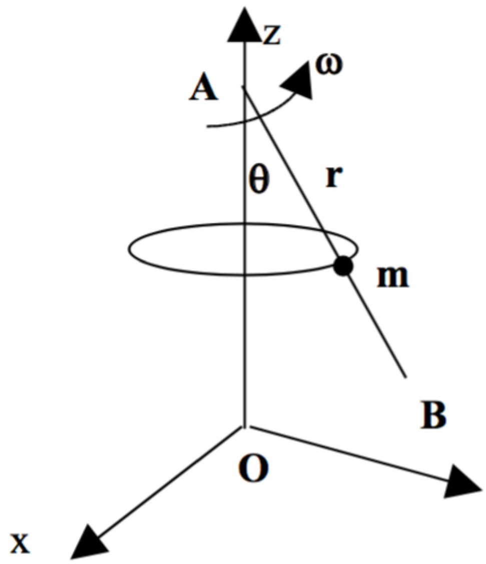
A positron e^+ with energy of 250 GeV/c^2 travels along the x-axis and collides with a stationary electron. A single particle V is produced, and only V remains after the collision. Later, V decays into two identical mass ($m = 0.1$ GeV/c^2) unstable muons μ^+ and μ^-, which have lifetimes of 2×10^{-6} s in their rest frame.
(a) Calculate v/c for the positron.
(b) What is the mass of particle V?
(c) What is the total energy of the particle V in its rest frame?
(d) What are the momenta of the electron and positron in the V rest frame?
(e) If the muon decays perpendicularly to the x axis in the V rest frame, what approximate angle does it make with respect to the x axis in the lab frame?
(f) How far would the muon travel in one lifetime as measured in the lab frame?

94. Small Oscillations [700 level]
Two pendulums are coupled by a massless spring with spring constant k. Both pendulums have massless springs of length L. They are separated by distance D. The masses are m and $2m$. Consider small oscillations.



(a) Find the normal modes of the pendulums.
(b) Determine the normal coordinates that undergo simple harmonic motion.

95. Lagrangian Mechanics [700 level]
A bead of mass m moves along a frictionless wire AB. The wire is fixed at point A and rotates with angular frequency ω about the z-axis. θ is fixed.



(a) Determine the Lagrangian in terms of r, θ, and the azimuthal angle.
(b) Determine the Euler-Lagrange equation.
(c) Solve the equation of motion.

96. Thermodynamics [500 level]
Consider a gas initially at temperature T_i, in a thermally isolated vessel of volume V_i. Suppose that a door is opened, allowing the gas to abruptly and freely expand into a final total volume V_f, without any work being done and without any heat transfer to the gas. At the end of this process, the gas reaches a final temperature T_f.
(a) In general, what is the change in internal energy as a result of this expansion?
(b) Suppose the gas is an ideal gas. What is the change in entropy of the gas as a result of the expansion? Explain.
(c) Now consider a non-ideal (interacting) gas. Show that the temperature change can be expressed as

$$\Delta T = -\int_{V_i}^{V_f} \frac{T^2}{C_V} \left(\frac{\partial (P/T)}{\partial T} \right)_V dV$$

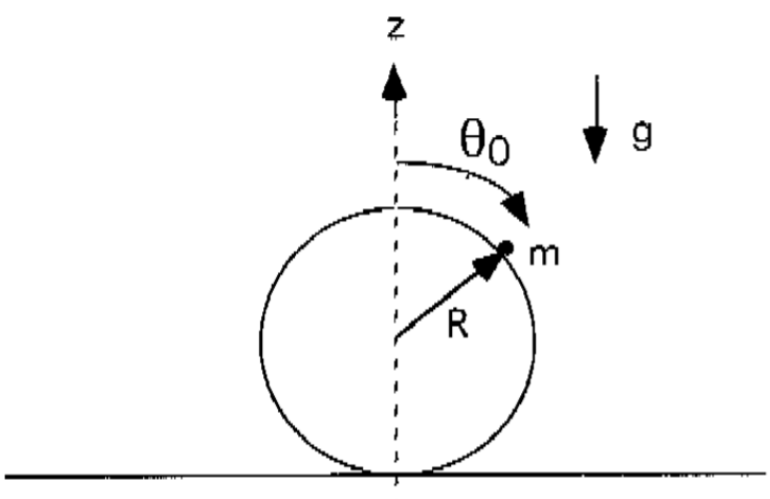
where the integration is done using the equilibrium equation of state of the gas. Make sure you justify the use of the equilibrium equation of state.
(d) Explain with justification whether in the general (non-ideal gas) case you expect the temperature change ΔT to be positive or negative.

97. Statistical Mechanics [500 level]
Consider an ideal monoatomic gas made of N atoms each of which has only

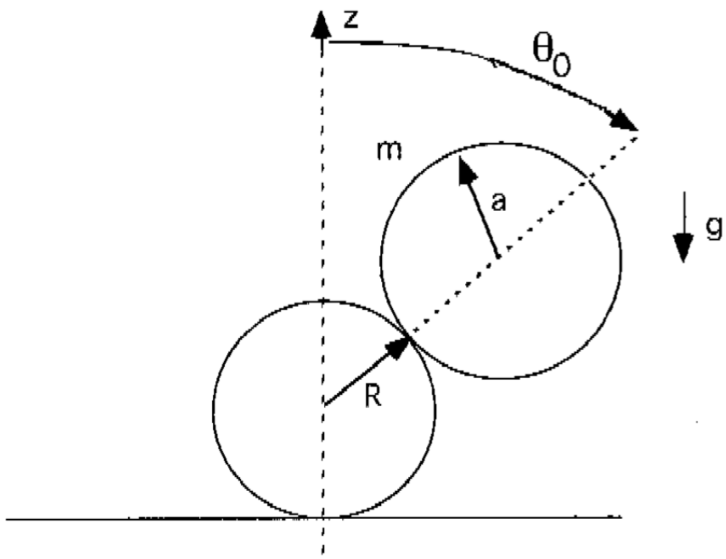
two internal states: a ground state and an excited state with energy gap equal to Δ. The gas is in a sealed container with no energy exchange with the outside world. Initially, the gas is prepared in such a way that all the atoms are in their ground state internally, but the gas is in thermal equilibrium with respect to kinetic motion of the atoms, characterized by temperature T_1. After some time, however, due to collisions, the internal degree of freedom of the atoms is also excited and thermalized.
(a) Find the temperature of the gas, T_2, after the internal degree of freedom thermalizes. Assume $\Delta \ll k_B T_1$, and calculate the difference $T_2 - T_1$ up to first order in Δ. Does the temperature increase or decrease?
(b) Assume $\Delta \ll k_B T_1$ and calculate the entropy change $S_2 - S_1$ up to first order in Δ after complete thermalization. Does the entropy increase or decrease? Hint: The entropy of the same gas *without* the internal degree of freedom is

$$S_{\text{kin}} = \frac{3}{2} k_B N \ln T + \text{terms independent of } T$$

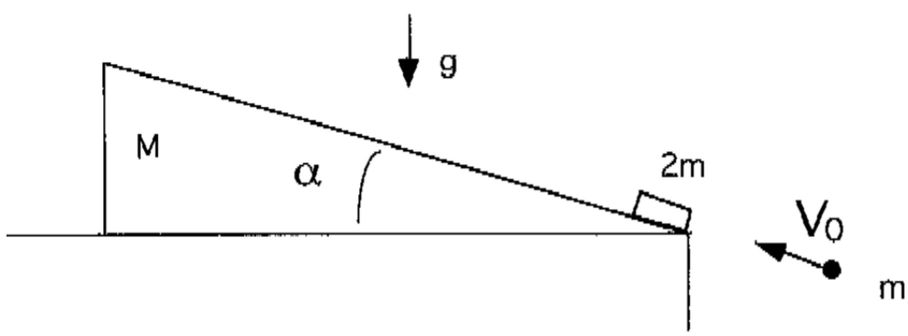
98. Newtonian Mechanics [500 level]
(a) At $t = 0$, a point particle of mass m is at rest on the frictionless surface of a fixed sphere. The sphere has radius R and the particle is initially at the angle θ_0 relative to the vertical z-axis through the center of the sphere. Gravity is directed as shown. After the particle is released, it leaves the surface of the sphere at angle θ_1. Find θ_1 in terms of θ_0.



(b) The point particle is replaced by a ball of mass m and radius a. The moment of inertia of the ball about its axis is $\frac{2}{5} m a^2$. The bottom sphere still cannot move, but there is now friction. The ball starts as before at an angle of θ_0 from the top of the sphere. Find the new θ_1 at which it leaves the surface. There is no slipping.

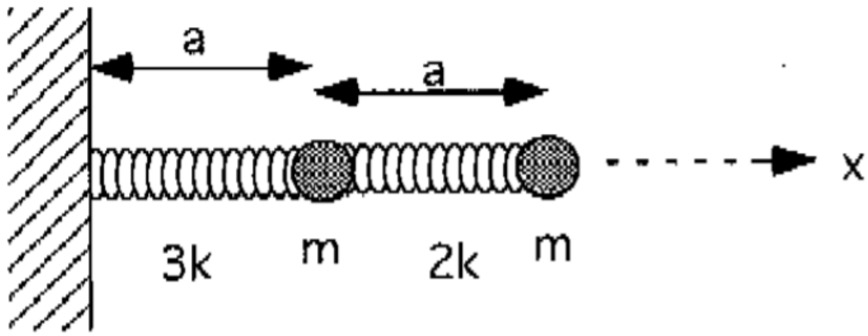


99. Newtonian Mechanics [500 level]
A slider of mass $2m$ is initially at rest at the bottom edge of a wedge of mass M and angle α. The wedge is completely frictionless, as is the table on which it sits. A bullet of mass m and velocity v_0 traveling parallel to the upper surface of the wedge hits and sticks to the slider.



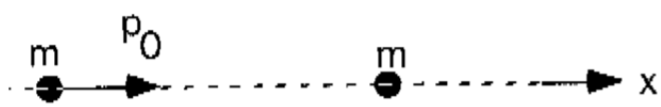
(a) What is the maximum height above the table reached by the slider?
(b) What is the wedge's velocity at that time?
(c) At what time does the slider reach maximum velocity?
(d) How far has the wedge moved by that time?

100. Lagrangian Mechanics [700 level]
Two objects, each mass m, are attached to each other and an immovable wall by springs, as in the figure, which shows the view from above. The masses are on a frictionless surface and can only move long the x-axis. The equilibrium length of both springs is a.

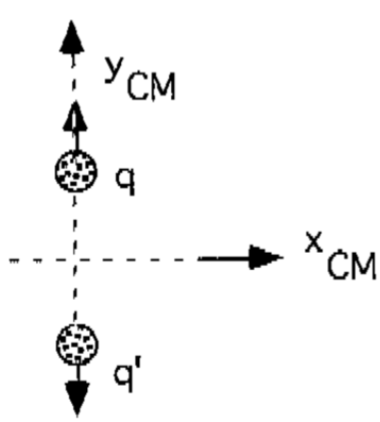


(a) Find the Lagrangian.
(b) Find the normal modes and their frequencies.
(c) At $t = 0$ the left mass is displaced from equilibrium a distance b to the left, and the right mass is displaced from equilibrium a distance b to the right. The masses are released with zero velocity. Find the positions of both masses as functions of time.

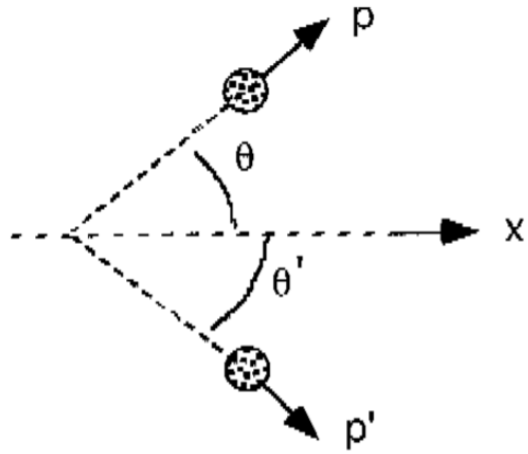
101. Special Relativity [500 level]
A particle of mass m moves long the x-axis with relativistic momentum p_0. It collides with another mass m particle that is stationary.



A reaction occurs, with the two mass m particles turning into two new particles, with masses m_1 and m_2, both of which are greater than m. After the collision, the two new particles move in opposite directions along the center of momentum frame y-axis. The mass m_1 particle moves in the positive $y_{\rm CM}$-direction with momentum $\mathbf{q}$, and the mass m_2 particle moves oppositely with momentum $\mathbf{q}'$. The lab frame x-axis and the CM frame x-axis are parallel.

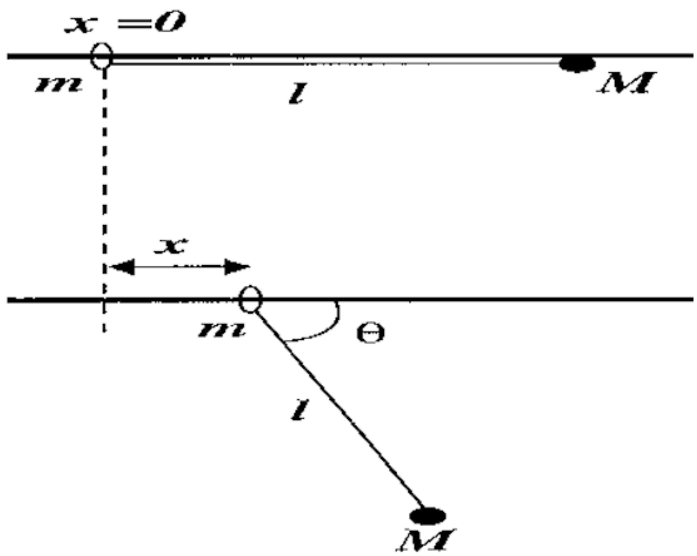


(a) How is $\mathbf{q}$ related to $\mathbf{q}'$?
(b) Find the minimum initial momentum, $p_{\min}$, required for this reaction to occur.
(c) If $p_0 > p_{\min}$, find the momentum magnitudes q and q'.
(d) In the lab frame, the final particles have momenta p and p', making angles θ and θ' with the x-axis.



For the case where $m_1 = m_2$, find the momentum magnitudes p and p'.
(e) For $m_1 = m_2$, find the lab-frame angles θ and θ'.

102. Lagrangian Mechanics [700 level]
A ring of mass m can slide freely on a smooth straight horizontal rod. A particle of mass m is attached to the ring by a massless string of length l. The particle is initially in contact with the rod, with the string taut. It is then released and falls under gravity.



(a) Show that the distance, x, of the ring form its initial position when the string makes an angle θ with the horizontal is

$$x = \frac{Ml(1 - \cos\theta)}{M + m}$$

(b) Using the initial position of the system as the zero of potential energy, construct the Lagrangian for the system and obtain the Euler-Lagrange equation of motion for the angle θ.
(c) Find an expression for the constant of motion for this system in terms of θ and $\dot{\theta}$.
(d) If $M = m$, calculate the tension in the string when $\theta = 30°$.

103. Oscillations [500 level]
Two identical objects of mass m are connected by a spring with constant k. With the masses constrained to move in one dimension, one of the masses is subject to a driving force $F(t) = F_0 \cos\omega t$, starting from rest at $t = 0$
(a) Find the normal modes of oscillation of the system.
(b) Find the motion of the mass that is not subject to the direct external force.

104. Accelerated Coordinate Systems [700 level]
A uniform spherical ball rolls without slipping and without rolling friction on a turntable that is rotating in the horizontal plane with frequency Ω. The ball moves in a circle of radius r around the center of the turntable. Find the angular velocity ω of the ball's motion around the center.

105. Damped Oscillations [500 level]
A blob of putty of mass m falls from a height h onto a massless platform

which is supported by a spring of constant k. A dashpot provides damping. The relaxation time of the putty is very short compared to that of the putty-plus-platform; so the putty instantaneously hits and sticks.
(a) Sketch the displacement of the platform as a function of time, under the given initial conditions, when the platform with the putty attached is critically damped.
(b) Determine the amount of damping so that, under the given initial conditions, the platform settles to its final position the most rapidly *without overshoot*.

106. Adiabatic Changes [700 level]
A mass m slides on a horizontal frictionless track. It is connected to a spring fastened to a wall. Initially, the amplitude of the oscillations is A and the spring constant is k_1. The spring constant then decreases adiabatically (as the spring wears out) at a constant rate, until the value k_2 is reached. What is the new amplitude?

107. Celestial Mechanics [700 level]
If the solar system were immersed in a uniformly dense spherical cloud of weakly interacting massive particles (WIMPs) then objects in the solar system would experience gravitational forces from both the sun and the cloud of WIMPs, such that
$$F_r = -\frac{k}{r^2} - br.$$
(a) Assume the extra force due to the WIMPs is very small (that is, $b \ll k/r^3$). Find the frequency of radial oscillations for a nearly circular orbit.
(b) Again assuming $b \ll k/r^3$, find the frequency of perihelion precession for a nearly circular orbit.
(c) Describe the shapes of the orbits when r is large enough that $F_r \approx -br$.

108. Negative Temperature [500 level]
Consider a system of $N \gg 1$ noninteracting particles in which the energy of each particle can assume two and only two values: 0 and E (with $E > 0$). Denote by n_0 and n_1 the occupation numbers of the energy levels 0 and E, respectively. The fixed energy of the system is U.
(a) Find the entropy S of the system.
(b) Find the temperature as a function of U. For what range of values is $T < 0$?
(c) In what direction does heat flow when a system of negative temperature is brought into contact with a system of positive temperature? Why?

109. Molecular Rotation [700 level]
Consider a heteronuclear diatomic molecule (such as HF) with moment of

inertia I. In this problem, only the rotational motion of the molecule should be considered.
(a) Using classical statistical mechanics, calculate the specific heat $C(T)$ of this system at temperature T.
(b) In quantum mechanics, this system has energy levels

$$E_j = \frac{\hbar^2}{2I}\ell(\ell+1),\ \ell = 0, 1, 2, \ldots.$$

Each energy level is $(2\ell+1)$-fold degenerate. Find expressions for the partition function Z and the average energy $\langle E \rangle$ of this system as a function of temperature. Do not attempt to evaluate these expressions.
(c) By simplifying your expressions in part (b), derive an expression for the specific heat $C(T)$ that is valid at very low temperatures. In what range of temperatures is your expression valid?
(d) By simplifying your expressions in part (b), derive an expression for the specific heat $C(T)$ that is valid at very high temperatures. In what range of temperatures is your expression valid?

110. Phase Separation [700 level]
The temperature of a long vertical column of a particular substance is T everywhere. Below a certain height $h(T)$, the substance is solid, whereas above $h(T)$ it is in a liquid phase. Calculate the density different between the phases, $\Delta\rho = \rho_s - \rho_l$ ($\Delta\rho \ll \rho_s$), in terms of: L (the latent heat of fusion per unit mass), T, dh/dT, and g (the acceleration of gravity).

111. Adsorption [700 level]
Consider a vapor (dilute gas) of number density n in equilibrium with a sub-monolayer (i.e., less than one atomic layer) adsorbed film. Model the binding of atoms to the surface with a potential energy $-\epsilon_0$. Assume there are N possible sites for adsorption per unit area, and find the pressure as a function of the surface concentration, n/N.

112. Adiabatic Changes [500 level]
A satellite is put into a circular orbit at a distance R_0 above the center of the Earth. A viscous force resulting from the thin upper atmosphere has a magnitude $F_v = Av^\alpha$, where v is the velocity of the satellite. It is observed that this results in a rate of change of the radial distance r given by $dr/dt = -C$, where C is a positive constant, sufficiently small so that the loss of energy per orbit is small compared to the total kinetic energy. Obtain expressions for A and α.

113. Gravitational Motion [500 level]
Three masses (m_1, m_2, and m_3), forming the corner of an equilateral triangle

of side a, attract each other gravitationally. Determine the form of rotational motion which will leave the *relative* positions of the three masses unchanged.

114. Torque [500 level]
A uniform thin rigid rod of weight W and length L is supported horizontally by two vertical props at its ends. At $t = 0$, one of these supports is kicked away. Find the force on the other support immediately thereafter.

115. Fluctuations [700 level]
An LC circuit is used as a thermometer by measuring the noise voltage across an inductor and capacitor in parallel. Find the relation between the rms noise voltage and the absolute temperature T.

116. Van der Waals Gas [500 level]
Calculate the Joule-Thomson coefficient $(\partial U/\partial V)_T$ for a van der Waals gas where

$$P = \frac{RT}{V - B} - \frac{a}{V^2},$$

for constants B and a.

117. Adiabatic Processes [500 level]
Consider a pressure-volume diagram for a given mass of a substance, on which there is a family of adiabatic curves. Prove that no two of these adiabatic curves can intersect.

118. Entropy [500 level]
Two identical perfect gasses with the same pressure P and the same number of particles N, but with different temperatures T_1 and T_2 are confined in two vessels, of volume V_1 and V_2, which are then connected together. Find the change in entropy after the system has reached equilibrium.

119. Statistical Equilibrium [500 level]
A rocket weighing 1000 kg is shot into deep space. Under the assumption that all stellar bodies have an average mass of 10^{30} kg each and move about with random speeds averaging 10 km/s, what is the average speed that the rocket will tend to assume after a very long time. (Neglect the possibility of the rocket falling into a star.)

120. Spin System [700 level]
Three particles at the corners of an equilateral triangle each carry a spin $\frac{1}{2}$. Their mutual Hamiltonian is given by

$$H = \frac{\lambda}{3}\left(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_3 + \boldsymbol{\sigma}_2 \cdot \boldsymbol{\sigma}_3\right).$$

(a) List the energy levels of this spin system, giving their total spin values and degeneracies.
(b) Deduce the partition function Z if the system is at temperature T.

121. Optical Doubles [500 level]
There are about 6500 stars visible to the naked eye. Sometimes two stars appear very close together, though upon careful examination no physical connection is found between them (i.e., they are at different distances). Such a pair is called an "optical double star."
(a) Assuming the stars to be distributed at random on the celestial sphere, compute the expectation value of the number of optical doubles with a separation of no more than $1'$ of arc.
(b) What is the probability that there are precisely two optical double stars?
(c) Estimate the probability of an optical triple star.