Honor Code Statement

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.
As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.
Furthermore, I have not only read but will also follow the instructions on the exam.

Signature : ______________________________

Name (printed) : ______________________________

INSTRUCTIONS:

(1) Start each new problem on a separate page.
(2) Write your name (or just your initials) and problem number on the top of each page.
(3) Write your solutions on only one side of your paper.
(4) When finished with the exam, put the problems in order and then consecutively number your pages.
(5) You have 3 hours for this exam but you may take 4 hours.
(6) Questions 1-8 are each worth 10 points. Question 9 is worth 20 points.

Notation:

- $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.
- Let $\mathbb{K}$ be the field of the real numbers $\mathbb{R}$ or of the extended real numbers $\widehat{\mathbb{R}}$. For $1 \leq p \leq \infty$, $L_p\left((X, \mathcal{F}, \mu) ; \mathbb{K}\right)$, or just L_p if confusion seems unlikely, denotes the space of (equivalence classes of) functions $f \colon X \to \mathbb{K}$ with finite $\|\cdot\|_p$-norm. Similarly, $L_0\left((X, \mathcal{F}, \mu) ; \mathbb{K}\right)$ denotes the space of (equivalence classes of) $\mathcal{F}$-measurable functions $f \colon X \to \mathbb{K}$.

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1. Let $\gamma \colon [0, 2\pi] \to \mathbb{C}$ be given by $\gamma(t) = 5\, e^{it}$. Compute

$$\int_\gamma \left[z e^{3/z} + \frac{\cos z}{z^2 (z-\pi)^3} \right] dz. \tag{1.1}$$

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2. Let $f \in H(\mathbb{C})$ be an entire function and $\operatorname{Im}(f(z)) \geq 0$ for each $z \in \mathbb{C}$. Show that f is constant.

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3. Let (X, d) be a metric space. Let K be a compact subset of X and C be a closed subset of X. Show that $K + C := \{k + c \in X \colon k \in K \text{ and } c \in C\}$ is closed in X.

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4. Let A and B be disjoint closed subsets of a metric space (X, d).
Construct a continuous function $f \colon X \to \mathbb{R}$ such that

$$\begin{aligned} f(a) &= {}^{+}1 \quad &&, \text{ if } a \in A \\ f(b) &= {}^{-}1 \quad &&, \text{ if } b \in B \\ {}^{-}1 < f(x) &< {}^{+}1 \quad &&, \text{ if } x \in X \setminus (A \cup B). \end{aligned}$$

You need to clearly show why your function f does all it needs to do.
Stated in short, constructively clearly prove Urysohn's Lemma.

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5. Let $(X, \mathcal{F})$ be a measurable space and $\mathcal{B}_Y$ be the Borel sets of a separable metric space (Y, d). Show that a function $f \colon X \to Y$ is $(\mathcal{F}, \mathcal{B}_Y)$-measurable if and only if, for each fixed $y \in Y$, the function $g_y \colon X \to \mathbb{R}$ given by

$$g_y(x) := d(y, f(x)) \tag{5.1}$$

is measurable. If you use the fact that Y is separable, be sure to mention where.

6. Let $(\mathbb{R}, \mathcal{L}, \mu)$ be the Lebesgue measure space on $\mathbb{R}$ and $f \in L_p((\mathbb{R}, \mathcal{L}, \mu); \mathbb{R})$ where $1 \leq p < \infty$. For a $y \in \mathbb{R}$, define $\tau_y f \in L_p$ by

$$(\tau_y f)(x) \;=\; f(x-y). \tag{6.1}$$

Show that

$$\lim_{y \to 0} \|f - \tau_y f\|_p = 0. \tag{6.2}$$

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7. Let $([0,1], \mathcal{L}, \mu)$ be the Lebesgue measure space on $[0,1]$.
Let $f \in L_1(([0,1], \mathcal{L}, \mu); \mathbb{R})$ satisfy

$$\int_0^1 e^{nx} f(x)\, d\mu(x) = 0 \quad , \quad \forall n \in \mathbb{N}_0. \tag{7.1}$$

Show that $f = 0$ μ-a.e..

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8. Let $(X, \mathcal{F}, \mu)$ be a finite positive measure space (so $\mu\colon \mathcal{F} \to [0, \infty)$) and $1 \leq p < \infty$.
Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $L_p((X, \mathcal{F}, \mu); \mathbb{R})$ that converges μ-a.e. to $f \in L_0((X, \mathcal{F}, \mu); \mathbb{R})$.
Let the set $\{|f_n|^p : n \in \mathbb{N}\}$ is uniformly integrable, i.e.,

$$\forall \epsilon > 0\ \exists \delta_\epsilon > 0 \text{ such that if } E \in \mathcal{F} \text{ and } \mu(E) < \delta_\epsilon \text{ then } \int_E |f_n|^p \, d\mu < \epsilon \text{ for each } n \in \mathbb{N}. \tag{8.1}$$

Show that $f \in L_p((X, \mathcal{F}, \mu); \mathbb{R})$ and $f_n \to f$ in L_p-norm.

Remark: you may use, without proving, Egoroff's Theorem provided you state Egoroff's Theorem as well as define each involved mode of converges.

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9. State whether the statement is true or false (0pt). Then either prove or give a counterexample (4pt).

9.a. Let (X, d) be a metric space. If A and B are closed subsets of X, then their Minkowski sum $A + B := \{a + b \in X : a \in A, b \in B\}$ is closed in X.

9.b. Let $(\mathbb{R}, \mathcal{L}, \mu)$ is the Lebesgue measure space on $\mathbb{R}$, $L_s((\mathbb{R}, \mathcal{L}, \mu); \mathbb{R}) := L_s$, and $1 \leq p < q < r \leq \infty$. Then $L_q \subset L_p + L_r$. (Recall $L_p + L_r := \{g + h : g \in L_p, h \in L_r\}$.)

9.c. If a Lebesgue measurable subset E of $[0,1]$ has Lebesgue measure one, then E is dense in $[0,1]$.

9.d. If a sequence converges in L_p-norm, where $1 \leq p < \infty$, then the sequence also converges in measure.

9.e. There exists an entire function whose real part is $u(x,y) = xy - x + y$. (If true, also constuct such an entire function.)