Qualifying Examination in Algebra

January 2013

Note! You must **prove** every assertion. Write your answers as **legibly** as you can on the blank sheets of paper provided. Write **complete** answers in **complete sentences**. Make sure that your **notation is defined**!

Use only **one side** of each sheet; start each problem on a **new sheet** of paper; and be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.

If some problem is incorrect, then give a counterexample. If some problem is vague, then be sure to explain your interpretation of the problem.

Each problem is worth 10 points.

1. Throughout this problem, d is a fixed positive integer and $\mathbb{Q}$ is the field of rational numbers. For each polynomial $f = a_0 + a_1 t + \cdots + a_n t^n$ in the polynomial ring $\mathbb{Q}[t]$ and each integer i with $0 \le i \le d-1$, let $N_i(f)$ be the sum $\sum\limits_{j \equiv i \mod d} a_j$, where j varies over all non-negative integers which are congruent to i mod d. Let I be the following set of polynomials f in $\mathbb{Q}[t]$:
$$I = \{f \in \mathbb{Q}[t] \mid N_0(f) = N_1(f) = N_2(f) = \cdots = N_{d-1}(f)\}.$$
Is I an ideal of $\mathbb{Q}[t]$? If no, then give an example. If yes, then
   (a) prove that I is an ideal,
   (b) give a generator of the ideal I, and
   (c) prove that your answer to (b) is correct.

2. Let G be a group and let g_1 and g_2 be elements of G with finite order. Does the product g_1g_2 have to have finite order? If no, give an example. If yes, give a proof.

3. Let G be a group and let g_1 and g_2 be elements of G. List some hypotheses which guarantee that a pretty formula holds which relates the order of g_1g_2, the order of g_1, and the order of g_2. List the hypotheses; list the formula; and prove the statement. (Make your statement be the most general statement that you can prove.)

4. (In this problem $\mathbb{Z}_n$ represents the quotient group $\frac{\mathbb{Z}}{n\mathbb{Z}}$ where $\mathbb{Z}$ is the group of integers under addition and $n\mathbb{Z}$ is the subgroup of $\mathbb{Z}$ which consists of all multiples of n.) List all of the subgroups of $\mathbb{Z}_4 \oplus \mathbb{Z}_8$. Make sure that each subgroup appears

in your list exactly one time. Which subgroups from your list are isomorphic to one another?

5. Let $\mathbb{R}$ be the group of real numbers under addition, $\mathbb{Z}$ be the subgroup of integers, and G be the quotient group $\frac{\mathbb{R}}{\mathbb{Z}}$. Let H be the subgroup of G which is generated by the element represented by the class of $\frac{1}{4}$. Your job is to describe $\frac{G}{H}$. That is, you are to exhibit a "nice group" N and an isomorphism $\frac{G}{H} \to N$. You must prove that your candidate for an isomorphism is indeed an isomorphism. A complete proof is expected; although this proof may be sophisticated (hence short).

6. State the main theorem about ruler and compass construction. This theorem has the form: the complex number α is constructible by ruler and compass if and only if some algebraic condition. What is the algebraic condition? Sketch a proof of the Theorem. It is not necessary to give every detail; but be sure to explain all of the highlights.

7. For each field F listed below, factor $f(x) = x^7 - 1$ into irreducible factors in $F[x]$.
   (a) $F = \mathbb{Q}$ (the field of rational numbers),
   (b) $F = \mathbb{Z}_7$ (the field with 7 elements),
   (c) $F = \mathbb{R}$ (the field of real numbers), and
   (d) $F = \mathbb{Z}_2$ (the field with 2 elements).